Singular functions in groupoid algebras

York, Semigroup seminar

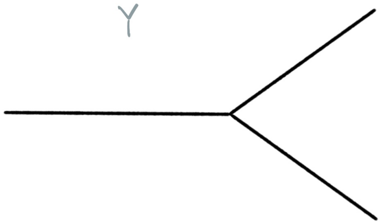
Nóra Szakács

Nov 26, 2025

Groupoids: a motivating example

Consider a group G acting on a locally compact, Hausdorff space Y.

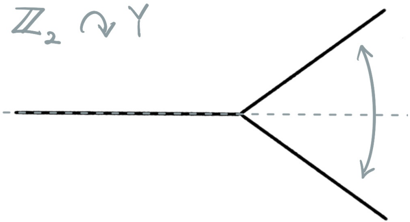
Example:



Groupoids: a motivating example

Consider a group G acting on a locally compact, Hausdorff space Y.

Example:

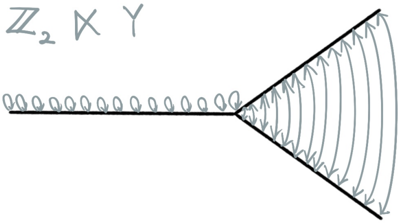


Groupoids: a motivating example

Consider a group G acting on a locally compact, Hausdorff space Y.

The associated transformation groupoid: $G \ltimes Y$

Example:



Convention: only the non-identity arrows are drawn

Groupoids: a motivating example

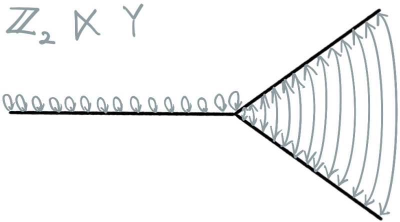
Consider a group G acting on a locally compact, Hausdorff space Y.

The associated transformation groupoid: $G \ltimes Y$

Objects: Y

Morphisms:
$\{(g, y) : y \in Y, g \in G\}$

Example:



Convention: only the non-identity arrows are drawn

Groupoids: a motivating example

Consider a group G acting on a locally compact, Hausdorff space Y.

The associated transformation groupoid: $G \ltimes Y$

Objects: Y

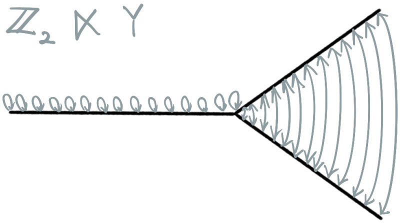
Morphisms:

$\{(g, y) : y \in Y, g \in G\}$

$s(g, y) = y$, $r(g, y) = g(y)$,

$(h, g(y))(g, y) = (hg, y)$

Example:



Convention: only the non-identity arrows are drawn

Groupoids: a motivating example

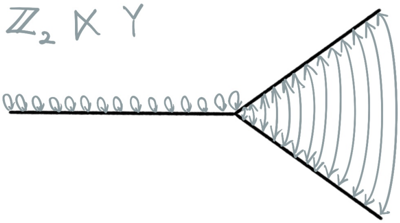
Consider a group G acting on a locally compact, Hausdorff space Y.

The associated transformation groupoid: $G \ltimes Y$

Objects: Y

Morphisms:
$\{(g, y) : y \in Y, g \in G\}$
$s(g, y) = y$, $r(g, y) = g(y)$,
$(h, g(y))(g, y) = (hg, y)$
We identify y and $(y, 1)$.

Example:



Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

Groupoids: a motivating example

Consider a group G acting on a locally compact, Hausdorff space Y.

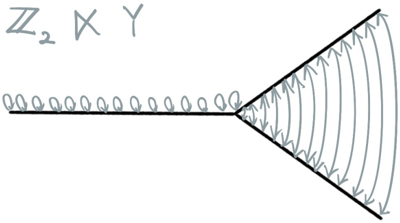
The associated transformation groupoid: $G \ltimes Y$

Objects: Y

Morphisms:
$\{(g, y) : y \in Y, g \in G\}$
$s(g, y) = y$, $r(g, y) = g(y)$,
$(h, g(y))(g, y) = (hg, y)$
We identify y and $(y, 1)$.

Topology:
sets of the form
$(g, U) = \{(g, y) : y \in U\}$
form a basis where
$g \in G, U \subseteq Y$ open

Example:



Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

Groupoids: a motivating example

Consider a group G acting on a locally compact, Hausdorff space Y.

The associated transformation groupoid: $G \ltimes Y$

Objects: Y

Morphisms:

$\{(g, y) : y \in Y, g \in G\}$

$s(g, y) = y$, $r(g, y) = g(y)$,

$(h, g(y))(g, y) = (hg, y)$

We identify y and $(y, 1)$.

Topology:

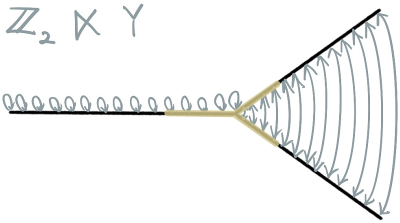
sets of the form

$(g, U) = \{(g, y) : y \in U\}$

form a basis where

$g \in G, U \subseteq Y$ open

Example:



Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

Groupoids: a motivating example

Consider a group G acting on a locally compact, Hausdorff space Y.

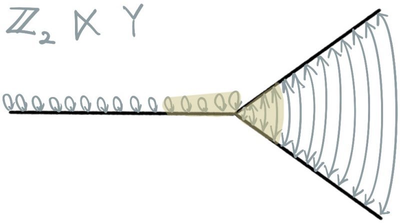
The associated transformation groupoid: $G \ltimes Y$

Objects: Y

Morphisms:
$\{(g, y) : y \in Y, g \in G\}$
$s(g, y) = y$, $r(g, y) = g(y)$,
$(h, g(y))(g, y) = (hg, y)$
We identify y and $(y, 1)$.

Topology:
sets of the form
$(g, U) = \{(g, y) : y \in U\}$
form a basis where
$g \in G, U \subseteq Y$ open

Example:

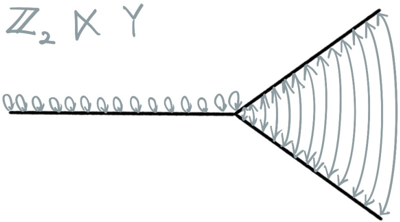


Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

Groupoids: a motivating example

We quotient $G \ltimes Y$ to obtain: the groupoid of germs

Example:



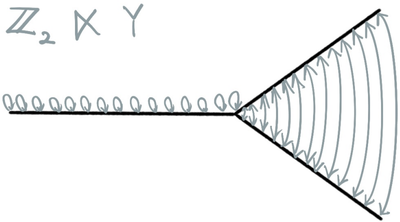
Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

Groupoids: a motivating example

We quotient $G \ltimes Y$ to obtain: the groupoid of germs

Define $(g, y) \sim (h, y)$ if there exists $U \subseteq Y$ open such that $y \in U$ and $g|_U = h|_U$

Example:



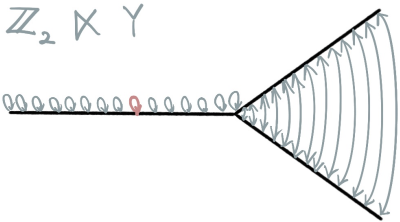
Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

Groupoids: a motivating example

We quotient $G \ltimes Y$ to obtain: the groupoid of germs

Define $(g, y) \sim (h, y)$ if there exists $U \subseteq Y$ open such that $y \in U$ and $g|_U = h|_U$

Example:



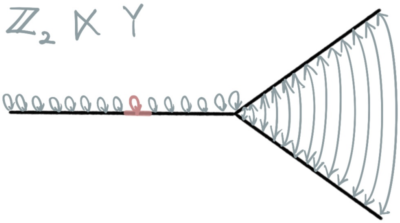
Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

Groupoids: a motivating example

We quotient $G \ltimes Y$ to obtain:
the groupoid of germs

Define $(g, y) \sim (h, y)$ if there exists $U \subseteq Y$ open such that $y \in U$ and $g|_U = h|_U$

Example:



Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

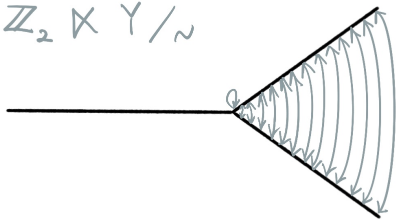
Groupoids: a motivating example

We quotient $G \ltimes Y$ to obtain: the groupoid of germs

Define $(g, y) \sim (h, y)$ if there exists $U \subseteq Y$ open such that $y \in U$ and $g|_U = h|_U$

$G \ltimes Y / \sim$ is a topological groupoid with the quotient topology

Example:



Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

Groupoids: a motivating example

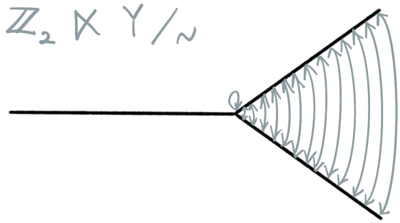
We quotient $G \ltimes Y$ to obtain: the groupoid of germs

Define $(g, y) \sim (h, y)$ if there exists $U \subseteq Y$ open such that $y \in U$ and $g|_U = h|_U$

$G \ltimes Y/\sim$ is a topological groupoid with the quotient topology

Basis:
$(g, U) = \{[g, y] : y \in U\}$ where $g \in G, U \subseteq Y$ open

Example:



Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

Groupoids: a motivating example

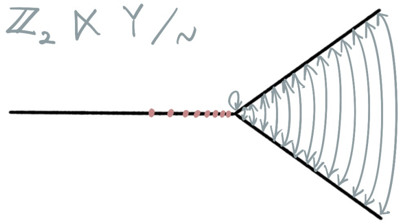
We quotient $G \ltimes Y$ to obtain:
the groupoid of germs

Define $(g, y) \sim (h, y)$ if there exists $U \subseteq Y$ open such that $y \in U$ and $g|_U = h|_U$

$G \ltimes Y / \sim$ is a topological groupoid with the quotient topology

Basis:
$(g, U) = \{[g, y] : y \in U\}$ where $g \in G, U \subseteq Y$ open

Example:



Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

Groupoids: a motivating example

We quotient $G \ltimes Y$ to obtain: the groupoid of germs

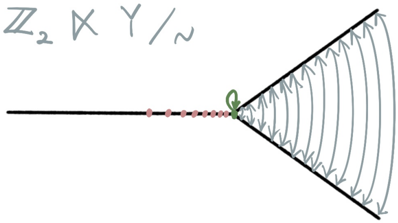
Define $(g, y) \sim (h, y)$ if there exists $U \subseteq Y$ open such that $y \in U$ and $g|_U = h|_U$

$G \ltimes Y/\sim$ is a topological groupoid with the quotient topology

Basis:

$(g, U) = \{[g, y] : y \in U\}$ where $g \in G, U \subseteq Y$ open

Example:



Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

Groupoids: a motivating example

We quotient $G \ltimes Y$ to obtain:
the groupoid of germs

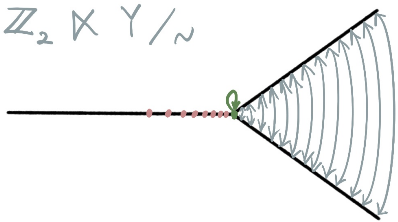
Define $(g, y) \sim (h, y)$ if there exists $U \subseteq Y$ open such that $y \in U$ and $g|_U = h|_U$

$G \ltimes Y/\sim$ is a topological groupoid with the quotient topology

Basis:
$(g, U) = \{[g, y] : y \in U\}$ where $g \in G, U \subseteq Y$ open

It is typically *not* a Hausdorff space!

Example:



Convention: only the non-identity arrows are drawn; the identity arrows are identified with the objects.

A remark

Suppose we have an inverse semigroup S acting on a (locally compact, Hausdorff) space Y, by homeomorphisms between open subsets.

A remark

Suppose we have an inverse semigroup S acting on a (locally compact, Hausdorff) space Y, by homeomorphisms between open subsets.

The "transformation groupoid" $S \ltimes Y$ is now only an inverse semigroupoid: if e, f are different idempotents acting on $y \in Y$, then (e, y), (f, y) are different idempotents in the submonoid of loops at y.

A remark

Suppose we have an inverse semigroup S acting on a (locally compact, Hausdorff) space Y, by homeomorphisms between open subsets.

The "transformation groupoid" $S \ltimes Y$ is now only an inverse semigroupoid: if e, f are different idempotents acting on $y \in Y$, then (e, y), (f, y) are different idempotents in the submonoid of loops at y.

But these are identified under germination: if $U = D(e) \cap D(f)$, then $e|_U = f|_U$.

A remark

Suppose we have an inverse semigroup S acting on a (locally compact, Hausdorff) space Y, by homeomorphisms between open subsets.

The "transformation groupoid" $S \ltimes Y$ is now only an inverse semigroupoid: if e, f are different idempotents acting on $y \in Y$, then (e, y), (f, y) are different idempotents in the submonoid of loops at y.

But these are identified under germination: if $U = D(e) \cap D(f)$, then $e|_U = f|_U$.

The quotient $S \ltimes Y/\sim$ is still a *groupoid* of germs.

Étale groupoids

Let $\mathcal{G}$ be a topological groupoid, i.e. a groupoid with a topology such that source and range maps, and the operations are continuous.

Convention: the identity arrows are identified with the objects and form the unit space $\mathcal{G}^{(0)}$.

Étale groupoids

Let $\mathcal{G}$ be a topological groupoid, i.e. a groupoid with a topology such that source and range maps, and the operations are continuous.

Convention: the identity arrows are identified with the objects and form the unit space $\mathcal{G}^{(0)}$.

$\mathcal{G}$ is étale if:

- $\mathcal{G}^{(0)}$ is a locally compact Hausdorff space

Étale groupoids

Let $\mathcal{G}$ be a topological groupoid, i.e. a groupoid with a topology such that source and range maps, and the operations are continuous.

Convention: the identity arrows are identified with the objects and form the unit space $\mathcal{G}^{(0)}$.

$\mathcal{G}$ is étale if:

- $\mathcal{G}^{(0)}$ is a locally compact Hausdorff space
- the source (s) and range (r) maps are local homeomorphisms

Étale groupoids

Let $\mathcal{G}$ be a topological groupoid, i.e. a groupoid with a topology such that source and range maps, and the operations are continuous.

Convention: the identity arrows are identified with the objects and form the unit space $\mathcal{G}^{(0)}$.

$\mathcal{G}$ is étale if:

- $\mathcal{G}^{(0)}$ is a locally compact Hausdorff space
- the source (s) and range (r) maps are local homeomorphisms

$U \subseteq \mathcal{G}$ is an open bisection if it is open and $s|_U, r|_U$ are homeomorphisms.

Étale groupoids

Let $\mathcal{G}$ be a topological groupoid, i.e. a groupoid with a topology such that source and range maps, and the operations are continuous.

Convention: the identity arrows are identified with the objects and form the unit space $\mathcal{G}^{(0)}$.

$\mathcal{G}$ is étale if:

- $\mathcal{G}^{(0)}$ is a locally compact Hausdorff space
- it has a basis of open bisections

$U \subseteq \mathcal{G}$ is an open bisection if it is open and $s|_U, r|_U$ are homeomorphisms.

Étale groupoids

Let $\mathcal{G}$ be a topological groupoid, i.e. a groupoid with a topology such that source and range maps, and the operations are continuous.

Convention: the identity arrows are identified with the objects and form the unit space $\mathcal{G}^{(0)}$.

$\mathcal{G}$ is étale if:

- $\mathcal{G}^{(0)}$ is a locally compact Hausdorff space
- it has a basis of open bisections

$U \subseteq \mathcal{G}$ is an open bisection if it is open and $s|_U, r|_U$ are homeomorphisms.

Examples: transformation groupoids, groupoids of germs

Étale groupoids

Let $\mathcal{G}$ be a topological groupoid, i.e. a groupoid with a topology such that source and range maps, and the operations are continuous.

Convention: the identity arrows are identified with the objects and form the unit space $\mathcal{G}^{(0)}$.

$\mathcal{G}$ is ample if:

- $\mathcal{G}^{(0)}$ is a locally compact Hausdorff space
- it has a basis of compact open bisections

$U \subseteq \mathcal{G}$ is an open bisection if it is open and $s|_U, r|_U$ are homeomorphisms.

Examples: transformation groupoids, groupoids of germs

Étale groupoids

Let $\mathcal{G}$ be a topological groupoid, i.e. a groupoid with a topology such that source and range maps, and the operations are continuous.

Convention: the identity arrows are identified with the objects and form the unit space $\mathcal{G}^{(0)}$.

$\mathcal{G}$ is ample if:

- $\mathcal{G}^{(0)}$ is a locally compact totally disconnected Hausdorff space
- it has a basis of open bisections

$U \subseteq \mathcal{G}$ is an open bisection if it is open and $s|_U, r|_U$ are homeomorphisms.

Examples: transformation groupoids, groupoids of germs

Groupoid algebras

Given an ample groupoid $\mathcal{G}$, its (complex) Steinberg algebra $\mathbb{C}\mathcal{G}$ consist of

$$\text{span}\{\chi_U : U \text{ is a compact open bisection}\} \subseteq \ell^\infty(\mathcal{G})$$

where $\ell^\infty(\mathcal{G})$ is the vector space of bounded $\mathcal{G} \to \mathbb{C}$ functions.

Groupoid algebras

Given an ample groupoid $\mathcal{G}$, its (complex) Steinberg algebra $\mathbb{C}\mathcal{G}$ consist of

$$\text{span}\{\chi_U : U \text{ is a compact open bisection}\} \subseteq \ell^\infty(\mathcal{G})$$

where $\ell^\infty(\mathcal{G})$ is the vector space of bounded $\mathcal{G} \to \mathbb{C}$ functions.

$\mathbb{C}\mathcal{G}$ is equipped with the convolution product:

$$\mathfrak{f} * \mathfrak{g}(\gamma) = \sum_{\gamma=\alpha\beta} \mathfrak{f}(\alpha)\mathfrak{g}(\beta).$$

Groupoid algebras

Given an ample groupoid $\mathcal{G}$, its (complex) Steinberg algebra $\mathbb{C}\mathcal{G}$ consist of

$$\text{span}\{\chi_U : U \text{ is a compact open bisection}\} \subseteq \ell^\infty(\mathcal{G})$$

where $\ell^\infty(\mathcal{G})$ is the vector space of bounded $\mathcal{G} \to \mathbb{C}$ functions.

$\mathbb{C}\mathcal{G}$ is equipped with the convolution product:

$$\mathfrak{f} * \mathfrak{g}(\gamma) = \sum_{\gamma=\alpha\beta} \mathfrak{f}(\alpha)\mathfrak{g}(\beta).$$

Example: let $\mathfrak{f} \in \mathbb{C}\mathcal{G}$ and $U \subseteq \mathcal{G}^{(0)}$ compact open.

Then

$$\mathfrak{f} * \chi_U(\gamma) =$$

Groupoid algebras

Given an ample groupoid $\mathcal{G}$, its (complex) Steinberg algebra $\mathbb{C}\mathcal{G}$ consist of

$$\mathrm{span}\{\chi_U : U \text{ is a compact open bisection}\} \subseteq \ell^\infty(\mathcal{G})$$

where $\ell^\infty(\mathcal{G})$ is the vector space of bounded $\mathcal{G} \to \mathbb{C}$ functions.

$\mathbb{C}\mathcal{G}$ is equipped with the convolution product:

$$\mathfrak{f} * \mathfrak{g}(\gamma) = \sum_{\gamma=\alpha\beta} \mathfrak{f}(\alpha)\mathfrak{g}(\beta).$$

Example: let $\mathfrak{f} \in \mathbb{C}\mathcal{G}$ and $U \subseteq \mathcal{G}^{(0)}$ compact open.

Then

$$\mathfrak{f} * \chi_U(\gamma) = \begin{cases} \mathfrak{f}(\gamma) & \text{if } s(\gamma) \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Example

If $\mathcal{G}$ is a discrete group G, then the compact open bisections are the singletons,

$$\mathbb{C}\mathcal{G} = \mathrm{span}\{\chi_g : g \in G\}$$

Example

If $\mathcal{G}$ is a discrete group G, then the compact open bisections are the singletons,

$$\mathbb{C}\mathcal{G} = \mathrm{span}\{\chi_g : g \in G\} \leftrightarrow \mathbb{C}G.$$

Example

If $\mathcal{G}$ is a discrete group G, then the compact open bisections are the singletons,

$$\mathbb{C}\mathcal{G} = \text{span}\{\chi_g : g \in G\} \leftrightarrow \mathbb{C}G.$$

$$\chi_g * \chi_h(k) = \sum_{k=k_1 k_2} \chi_g(k_1)\chi_h(k_2) =$$

Example

If $\mathcal{G}$ is a discrete group G, then the compact open bisections are the singletons,

$$\mathbb{C}\mathcal{G} = \mathrm{span}\{\chi_g : g \in G\} \leftrightarrow \mathbb{C}G.$$

$$\chi_g * \chi_h(k) = \sum_{k=k_1 k_2} \chi_g(k_1)\chi_h(k_2) = \begin{cases} 1 & \text{if } k = gh, \\ 0 & \text{otherwise.} \end{cases}$$

Example

If $\mathcal{G}$ is a discrete group G, then the compact open bisections are the singletons,

$$\mathbb{C}\mathcal{G} = \text{span}\{\chi_g : g \in G\} \leftrightarrow \mathbb{C}G.$$

$$\chi_g * \chi_h(k) = \sum_{k=k_1k_2} \chi_g(k_1)\chi_h(k_2) = \begin{cases} 1 & \text{if } k = gh, \\ 0 & \text{otherwise.} \end{cases}$$

So $\chi_g * \chi_h = \chi_{gh}$ and $\mathbb{C}\mathcal{G} \cong \mathbb{C}G$.

Example

If $\mathcal{G}$ is a discrete group G, then the compact open bisections are the singletons,

$$\mathbb{C}\mathcal{G} = \text{span}\{\chi_g : g \in G\} \leftrightarrow \mathbb{C}G.$$

$$\chi_g * \chi_h(k) = \sum_{k=k_1k_2} \chi_g(k_1)\chi_h(k_2) = \begin{cases} 1 & \text{if } k = gh, \\ 0 & \text{otherwise.} \end{cases}$$

So $\chi_g * \chi_h = \chi_{gh}$ and $\mathbb{C}\mathcal{G} \cong \mathbb{C}G$.

Remark: inverse semigroup algebras are also isomorphic to some $\mathbb{C}\mathcal{G}$ (Steinberg [Adv. Math, 2010]), but here $\mathcal{G}$ is usually not Hausdorff.

Primer on C^*-algebras

We will also associate C^*-algebras to groupoids. But what is a C^*-algebra?

Primer on C^*-algebras

We will also associate C^*-algebras to groupoids. But what is a C^*-algebra?

It is a:

- complex algebra
- with an involutive operation $()^*$ $(c\mathfrak{a}+\mathfrak{b})^* = \overline{c}\mathfrak{a}^* + \mathfrak{b}^*$, $(ab)^* = b^*a^*$
- and a complete norm $||\cdot||$ $||\mathfrak{a}\mathfrak{b}|| \leq ||\mathfrak{a}|| \cdot ||\mathfrak{b}||$, $||\mathfrak{a}\mathfrak{a}^*|| = ||\mathfrak{a}||^2$

Primer on C^*-algebras

We will also associate C^*-algebras to groupoids. But what is a C^*-algebra?

It is a:

- complex algebra
- with an involutive operation $()^*$ $(c\mathfrak{a}+\mathfrak{b})^* = \overline{c}\mathfrak{a}^* + \mathfrak{b}^*$, $(ab)^* = b^*a^*$
- and a complete norm $||\cdot||$ $||\mathfrak{a}\mathfrak{b}|| \leq ||\mathfrak{a}|| \cdot ||\mathfrak{b}||$, $||\mathfrak{a}\mathfrak{a}^*|| = ||\mathfrak{a}||^2$

Examples:

1. full complex matrix algebras
2. more generally: bounded linear operators on a Hilbert space: $\mathcal{B}(\mathcal{H})$

Groupoid C*-algebras

The reduced C^*-algebra of an ample groupoid $\mathcal{G}$ is the completion of $\mathbb{C}\mathcal{G}$ in reduced norm.

Groupoid C*-algebras

The reduced C^*-algebra of an ample groupoid $\mathcal{G}$ is the completion of $\mathbb{C}\mathcal{G}$ in reduced norm.

For $x \in \mathcal{G}^{(0)}$, let $\mathcal{G}_x = \{\gamma \in \mathcal{G} : s(\gamma) = x\}$, and consider the family of representations

$$\lambda_x : \mathbb{C}\mathcal{G} \to \mathcal{B}(\ell^2(\mathcal{G}_x))$$
$$\lambda_x(\mathfrak{f})(\delta_\gamma) = \sum_{\alpha : s(\alpha) = r(\gamma)} \mathfrak{f}(\alpha)\delta_{\alpha\gamma}.$$

We define the reduced norm of $\mathfrak{f}$ as $||\mathfrak{f}|| = \sup_{x \in \mathcal{G}^{(0)}} ||\lambda_x(\mathfrak{f})||$.

The reduced C^*-algebra of $\mathcal{G}$: $C^*_{red}(\mathcal{G}) := \overline{\mathbb{C}\mathcal{G}}^{||\cdot||}$.

Groupoid C*-algebras

The reduced C^*-algebra of an ample groupoid $\mathcal{G}$ is the completion of $\mathbb{C}\mathcal{G}$ in reduced norm.

For $x \in \mathcal{G}^{(0)}$, let $\mathcal{G}_x = \{\gamma \in \mathcal{G} : s(\gamma) = x\}$, and consider the family of representations

$$\lambda_x \colon \mathbb{C}\mathcal{G} \to \mathcal{B}(\ell^2(\mathcal{G}_x))$$
$$\lambda_x(\mathfrak{f})(\delta_\gamma) = \sum_{\alpha : s(\alpha) = r(\gamma)} \mathfrak{f}(\alpha)\delta_{\alpha\gamma}.$$

We define the reduced norm of $\mathfrak{f}$ as $||\mathfrak{f}|| = \sup_{x \in \mathcal{G}^{(0)}} ||\lambda_x(\mathfrak{f})||$.

The reduced C^*-algebra of $\mathcal{G}$: $C^*_{red}(\mathcal{G}) := \overline{\mathbb{C}\mathcal{G}}^{||\cdot||}$.

If $\mathcal{G}$ is a group or an inverse semigroup, then $C^*_{red}(\mathcal{G})$ coincides with the reduced group/inverse semigroup C^*-algebra as usually defined.

More examples

If X is a finite set with $|X| = n$, there is an ample groupoid $\mathcal{G}_n$ such that

$\mathbb{C}\mathcal{G}_n \cong L_{\mathbb{C}}(1, n)$ is the (complex) Leavitt algebra

$C^*_{red}(\mathcal{G}_n) \cong \mathcal{O}_n$ is the Cuntz algebra

More examples

If X is a finite set with $|X| = n$, there is an ample groupoid $\mathcal{G}_n$ such that

$$\mathbb{C}\mathcal{G}_n \cong L_{\mathbb{C}}(1, n) \text{ is the (complex) Leavitt algebra}$$

$$C^*_{red}(\mathcal{G}_n) \cong \mathcal{O}_n \text{ is the Cuntz algebra}$$

More generally:

If E is a directed graph, there is an ample groupoid $\mathcal{G}_E$ such that

$$\mathbb{C}\mathcal{G}_E \cong L_{\mathbb{C}}(E) \text{ is the (complex) Leavitt path algebra}$$

$$C^*_{red}(\mathcal{G}_E) \cong C^*(E) \text{ is the graph C*-algebra}$$

More examples

If X is a finite set with $|X| = n$, there is an ample groupoid $\mathcal{G}_n$ such that

$$\mathbb{C}\mathcal{G}_n \cong L_{\mathbb{C}}(1, n) \text{ is the (complex) Leavitt algebra}$$

$$C^*_{red}(\mathcal{G}_n) \cong \mathcal{O}_n \text{ is the Cuntz algebra}$$

More generally:

If E is a directed graph, there is an ample groupoid $\mathcal{G}_E$ such that

$$\mathbb{C}\mathcal{G}_E \cong L_{\mathbb{C}}(E) \text{ is the (complex) Leavitt path algebra}$$

$$C^*_{red}(\mathcal{G}_E) \cong C^*(E) \text{ is the graph C*-algebra}$$

Many ring-theoretic properties of the two algebras coincide: simplicity, purely infinite simplicity, primitivity, primeness...

The j-map

Recall: $C^*_{red}(\mathcal{G})$ is the completion of $\mathbb{C}\mathcal{G} \subseteq \ell^\infty(\mathcal{G})$ in norm.

The j-map

Recall: $C^*_{red}(\mathcal{G})$ is the completion of $\mathbb{C}\mathcal{G} \subseteq \ell^\infty(\mathcal{G})$ in norm.

Renault's j-map allows us to think of $C^*_{red}(\mathcal{G})$ as $\mathcal{G} \to \mathbb{C}$ functions.

The j-map

Recall: $C^*_{red}(\mathcal{G})$ is the completion of $\mathbb{C}\mathcal{G} \subseteq \ell^\infty(\mathcal{G})$ in norm.

Renault's j-map allows us to think of $C^*_{red}(\mathcal{G})$ as $\mathcal{G} \to \mathbb{C}$ functions.

Theorem (Renault, '80)
There exists an injective linear map $j \colon C^*_{red}(\mathcal{G}) \to \ell^\infty(\mathcal{G})$ such that for any $\mathfrak{a}, \mathfrak{b} \in C^*_{red}(\mathcal{G})$,

1. $j|_{\mathbb{C}\mathcal{G}}$ *is the identity;*
2. $j(\mathfrak{a}\mathfrak{b}) = \mathfrak{a} * \mathfrak{b}$*;*
3. $||\mathfrak{a}|| \geq ||j(\mathfrak{a})||_\infty$.

The j-map

Recall: $C^*_{red}(\mathcal{G})$ is the completion of $\mathbb{C}\mathcal{G} \subseteq \ell^\infty(\mathcal{G})$ in norm.

Renault's j-map allows us to think of $C^*_{red}(\mathcal{G})$ as $\mathcal{G} \to \mathbb{C}$ functions.

Theorem (Renault, '80)
There exists an injective linear map $j\colon C^*_{red}(\mathcal{G}) \to \ell^\infty(\mathcal{G})$ such that for any $\mathfrak{a}, \mathfrak{b} \in C^*_{red}(\mathcal{G})$,

1. $j|_{\mathbb{C}\mathcal{G}}$ *is the identity;*
2. $j(\mathfrak{a}\mathfrak{b}) = \mathfrak{a} * \mathfrak{b}$*;*
3. $||\mathfrak{a}|| \geq ||j(\mathfrak{a})||_\infty$.

Note: 3. says that if $\mathfrak{a}_n \xrightarrow{||\cdot||} \mathfrak{a}$, then $j(\mathfrak{a}_n) \xrightarrow{||\cdot||_\infty} j(\mathfrak{a})$.

The j-map

Recall: $C^*_{red}(\mathcal{G})$ is the completion of $\mathbb{C}\mathcal{G} \subseteq \ell^\infty(\mathcal{G})$ in norm.

Renault's j-map allows us to think of $C^*_{red}(\mathcal{G})$ as $\mathcal{G} \to \mathbb{C}$ functions.

Theorem (Renault, '80)
There exists an injective linear map $j\colon C^*_{red}(\mathcal{G}) \to \ell^\infty(\mathcal{G})$ such that for any $\mathfrak{a}, \mathfrak{b} \in C^*_{red}(\mathcal{G})$,

1. $j|_{\mathbb{C}\mathcal{G}}$ *is the identity;*
2. $j(\mathfrak{a}\mathfrak{b}) = \mathfrak{a} * \mathfrak{b}$*;*
3. $||\mathfrak{a}|| \geq ||j(\mathfrak{a})||_\infty$*.*

Note: 3. says that if $\mathfrak{a}_n \xrightarrow{||\cdot||} \mathfrak{a}$, then $j(\mathfrak{a}_n) \xrightarrow{||\cdot||_\infty} j(\mathfrak{a})$. In particular $j(C^*_{red}(\mathcal{G})) \subseteq \overline{\mathbb{C}\mathcal{G}}^{||\cdot||_\infty}$.

The j-map

Recall: $C^*_{red}(\mathcal{G})$ is the completion of $\mathbb{C}\mathcal{G} \subseteq \ell^\infty(\mathcal{G})$ in norm.

Renault's j-map allows us to think of $C^*_{red}(\mathcal{G})$ as $\mathcal{G} \to \mathbb{C}$ functions.

Theorem (Renault, '80)
There exists an injective linear map $j\colon C^*_{red}(\mathcal{G}) \to \ell^\infty(\mathcal{G})$ such that for any $\mathfrak{a}, \mathfrak{b} \in C^*_{red}(\mathcal{G})$,

1. $j|_{\mathbb{C}\mathcal{G}}$ *is the identity;*
2. $j(\mathfrak{a}\mathfrak{b}) = \mathfrak{a} * \mathfrak{b}$*;*
3. $||\mathfrak{a}|| \geq ||j(\mathfrak{a})||_\infty$.

Note: 3. says that if $\mathfrak{a}_n \xrightarrow{||\cdot||} \mathfrak{a}$, then $j(\mathfrak{a}_n) \xrightarrow{||\cdot||_\infty} j(\mathfrak{a})$. In particular $j(C^*_{red}(\mathcal{G})) \subseteq \overline{\mathbb{C}\mathcal{G}}^{||\cdot||_\infty}$.

From now on we identify $\mathfrak{a}$ with $j(\mathfrak{a})$ in notation.

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Hausdorff, second countable case:

Brown-Clark-Farthing-Sims [Semigroup Forum, 2014]

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Hausdorff, second countable case:

Brown-Clark-Farthing-Sims [Semigroup Forum, 2014]

$\iff$ $\mathcal{G}$ is minimal and effective

$\Longleftarrow$ $\mathcal{G}$ is minimal and effective
$\Longrightarrow$ $\mathcal{G}$ is minimal, effective when $\mathcal{G}$ is amenable

Minimal: every orbit is dense in the unit space, i.e. for all $x \in \mathcal{G}^{(0)}$, $\mathcal{G}x\mathcal{G} \cap \mathcal{G}^{(0)}$ is dense in $\mathcal{G}^{(0)}$.

Effective: $\operatorname{int}(\{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma)\}) = \mathcal{G}^{(0)}$

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Hausdorff, second countable case:

Brown-Clark-Farthing-Sims [Semigroup Forum, 2014]

$\iff$ $\mathcal{G}$ is minimal and effective

$\Longleftarrow$ $\mathcal{G}$ is minimal and effective
$\Longrightarrow$ $\mathcal{G}$ is minimal, effective when $\mathcal{G}$ is amenable

Minimal: every orbit is dense in the unit space, i.e. for all $x \in \mathcal{G}^{(0)}$, $\mathcal{G}x\mathcal{G} \cap \mathcal{G}^{(0)}$ is dense in $\mathcal{G}^{(0)}$.

Effective: $\operatorname{int}(\{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma)\}) = \mathcal{G}^{(0)}$

Remark: any groupoid of germs (as we defined it) is effective

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Non-Hausdorff, second countable case:

Clark-Exel-Pardo-Starling-Sims [Trans. AMS, 2019]

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Non-Hausdorff, second countable case:

Clark-Exel-Pardo-Starling-Sims [Trans. AMS, 2019]

Extended to non-second countable:

Steinberg-Sz. [Adv. Math, 2021]

Extended to non-second countable:

Kwaśniewski-Meyer [Doc. Math, 2021]

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Non-Hausdorff, second countable case:

Clark-Exel-Pardo-Starling-Sims [Trans. AMS, 2019]

Extended to non-second countable:

Steinberg-Sz. [Adv. Math, 2021]

$\iff$ $\mathcal{G}$ is minimal, effective, and $J_{alg} = 0$

Extended to non-second countable:

Kwaśniewski-Meyer [Doc. Math, 2021]

$\Longleftarrow$ $\mathcal{G}$ is minimal, effective, $J = 0$

$\Longrightarrow$ $\mathcal{G}$ is minimal, effective and $J = 0$ when $\mathcal{G}$ is amenable

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Non-Hausdorff, second countable case:

Clark-Exel-Pardo-Starling-Sims [Trans. AMS, 2019]

Extended to non-second countable:

Steinberg-Sz. [Adv. Math, 2021]

$\iff$ $\mathcal{G}$ is minimal, effective, and $J_{alg} = 0$

Extended to non-second countable:

Kwaśniewski-Meyer [Doc. Math, 2021]

$\Longleftarrow$ $\mathcal{G}$ is minimal, effective, $J = 0$

$\Longrightarrow$ $\mathcal{G}$ is minimal, effective and $J = 0$ when $\mathcal{G}$ is amenable

Where $J = \{\mathfrak{f} \in C^*_{red}(\mathcal{G}) : \mathfrak{f}$ vanishes on a dense subset$\}$ is called the (C^*-)singular ideal of $C^*_{red}(\mathcal{G})$.

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Non-Hausdorff, second countable case:

Clark-Exel-Pardo-Starling-Sims [Trans. AMS, 2019]

Extended to non-second countable:

Steinberg-Sz. [Adv. Math, 2021]

$\iff$ $\mathcal{G}$ is minimal, effective, and $J_{alg} = 0$

Extended to non-second countable:

Kwaśniewski-Meyer [Doc. Math, 2021]

$\Longleftarrow$ $\mathcal{G}$ is minimal, effective, $J = 0$

$\Longrightarrow$ $\mathcal{G}$ is minimal, effective and $J = 0$ when $\mathcal{G}$ is amenable

Where $J = \{\mathfrak{f} \in C^*_{red}(\mathcal{G}) : \mathfrak{f}$ vanishes on a dense subset$\}$ is called the (C^*-)singular ideal of $C^*_{red}(\mathcal{G})$.

$J_{alg} := J \cap \mathbb{C}\mathcal{G}$ is called the (algebraic) singular ideal of $\mathbb{C}\mathcal{G}$.

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Non-Hausdorff, second countable case:

Clark-Exel-Pardo-Starling-Sims [Trans. AMS, 2019]

Extended to non-second countable:

Steinberg-Sz. [Adv. Math, 2021]

$\iff$ $\mathcal{G}$ is minimal, effective, and $J_{alg} = 0$

Extended to non-second countable:

Kwaśniewski-Meyer [Doc. Math, 2021]

$\Longleftarrow$ $\mathcal{G}$ is minimal, effective, $J = 0$

$\Longrightarrow$ $\mathcal{G}$ is minimal, effective and $J = 0$ when $\mathcal{G}$ is amenable

Where $J = \{\mathfrak{f} \in C^*_{red}(\mathcal{G}) : \mathfrak{f}$ vanishes on a dense subset$\}$ is called the (C^*-)singular ideal of $C^*_{red}(\mathcal{G})$.

$J_{alg} := J \cap \mathbb{C}\mathcal{G}$ is called the (algebraic) singular ideal of $\mathbb{C}\mathcal{G}$.

$C^*_{red}(\mathcal{G})/J = C^*_{ess}(\mathcal{G})$ is the essential algebra of $\mathcal{G}$.

The singular ideal

Recall:

$\mathbb{C}\mathcal{G} = \mathrm{span}\{\chi_U : U \text{ is a compact open bisection}\}$, $C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}}$;

$J_{alg} = \{\mathfrak{f} \in \mathbb{C}\mathcal{G} : \mathfrak{f} \text{ vanishes on a dense subset}\} \lhd \mathbb{C}\mathcal{G}$.

The singular ideal

Recall:

$\mathbb{C}\mathcal{G} = \text{span}\{\chi_U : U \text{ is a compact open bisection}\}$, $C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}}$;

$J_{alg} = \{\mathfrak{f} \in \mathbb{C}\mathcal{G} : \mathfrak{f} \text{ vanishes on a dense subset}\} \lhd \mathbb{C}\mathcal{G}$.

Note: $\mathfrak{f} \in J \iff \text{supp}\,\mathfrak{f} = \{\gamma \in \mathcal{G} : \mathfrak{f}(\gamma) \neq 0\}$ contains no open set.

The singular ideal

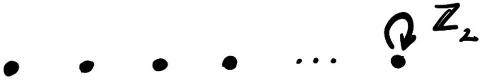
Recall:

$\mathbb{C}\mathcal{G} = \text{span}\{\chi_U : U \text{ is a compact open bisection}\}$, $C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}}$;

$J_{alg} = \{\mathfrak{f} \in \mathbb{C}\mathcal{G} : \mathfrak{f} \text{ vanishes on a dense subset}\} \lhd \mathbb{C}\mathcal{G}$.

Note: $\mathfrak{f} \in J \iff \operatorname{supp}\mathfrak{f} = \{\gamma \in \mathcal{G} : \mathfrak{f}(\gamma) \neq 0\}$ contains no open set.

Example:



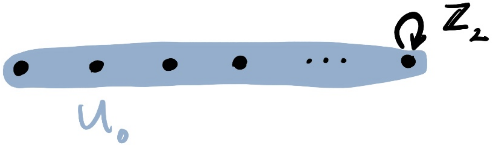
Recall:

$\mathbb{C}\mathcal{G} = \operatorname{span}\{\chi_U : U \text{ is a compact open bisection}\}$, $C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}}$;

$J_{alg} = \{\mathfrak{f} \in \mathbb{C}\mathcal{G} : \mathfrak{f} \text{ vanishes on a dense subset}\} \lhd \mathbb{C}\mathcal{G}$.

Note: $\mathfrak{f} \in J \iff \operatorname{supp} \mathfrak{f} = \{\gamma \in \mathcal{G} : \mathfrak{f}(\gamma) \neq 0\}$ contains no open set.

Example:



The singular ideal

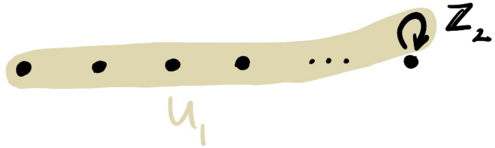
Recall:

$\mathbb{C}\mathcal{G} = \text{span}\{\chi_U : U \text{ is a compact open bisection}\}$, $C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}}$;

$J_{alg} = \{\mathfrak{f} \in \mathbb{C}\mathcal{G} : \mathfrak{f} \text{ vanishes on a dense subset}\} \lhd \mathbb{C}\mathcal{G}$.

Note: $\mathfrak{f} \in J \iff \text{supp}\,\mathfrak{f} = \{\gamma \in \mathcal{G} : \mathfrak{f}(\gamma) \neq 0\}$ contains no open set.

Example:



The singular ideal

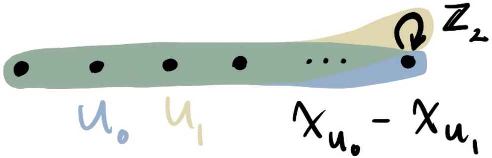
Recall:

$\mathbb{C}\mathcal{G} = \mathrm{span}\{\chi_U : U \text{ is a compact open bisection}\}$, $C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}}$;

$J_{alg} = \{\mathfrak{f} \in \mathbb{C}\mathcal{G} : \mathfrak{f} \text{ vanishes on a dense subset}\} \lhd \mathbb{C}\mathcal{G}$.

Note: $\mathfrak{f} \in J \iff \mathrm{supp}\,\mathfrak{f} = \{\gamma \in \mathcal{G} : \mathfrak{f}(\gamma) \neq 0\}$ contains no open set.

Example:



The singular ideal

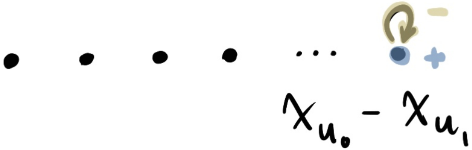
Recall:

$\mathbb{C}\mathcal{G} = \text{span}\{\chi_U : U \text{ is a compact open bisection}\}$, $C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}}$;

$J_{alg} = \{\mathfrak{f} \in \mathbb{C}\mathcal{G} : \mathfrak{f} \text{ vanishes on a dense subset}\} \lhd \mathbb{C}\mathcal{G}$.

Note: $\mathfrak{f} \in J \iff \operatorname{supp}\mathfrak{f} = \{\gamma \in \mathcal{G} : \mathfrak{f}(\gamma) \neq 0\}$ contains no open set.

Example:



The singular ideal

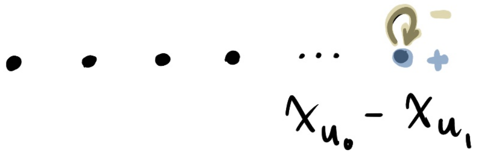
Recall:

$\mathbb{C}\mathcal{G} = \text{span}\{\chi_U : U \text{ is a compact open bisection}\}$, $C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}}$;

$J_{alg} = \{\mathfrak{f} \in \mathbb{C}\mathcal{G} : \mathfrak{f} \text{ vanishes on a dense subset}\} \lhd \mathbb{C}\mathcal{G}$.

Note: $\mathfrak{f} \in J \iff \text{supp}\,\mathfrak{f} = \{\gamma \in \mathcal{G} : \mathfrak{f}(\gamma) \neq 0\}$ contains no open set.

Example:



$$\chi_{U_0} - \chi_{U_1} \in J_{alg}$$

The singular ideal – a minimal and effective example

In Clark-Exel-Pardo-Starling-Sims [Trans. AMS, 2019], it is open question whether $J \neq 0$ can happen in minimal and effective groupoids.

The singular ideal – a minimal and effective example

In Clark-Exel-Pardo-Starling-Sims [Trans. AMS, 2019], it is open question whether $J \neq 0$ can happen in minimal and effective groupoids.

In fact Nekrashevych had already been aware of such an example (coming from the Grigorchuk-Erschler self-similar group).

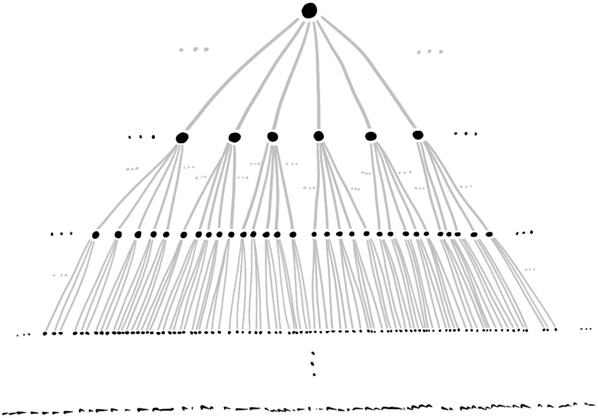
The singular ideal – a minimal and effective example

In Clark-Exel-Pardo-Starling-Sims [Trans. AMS, 2019], it is open question whether $J \neq 0$ can happen in minimal and effective groupoids.

In fact Nekrashevych had already been aware of such an example (coming from the Grigorchuk-Erschler self-similar group).

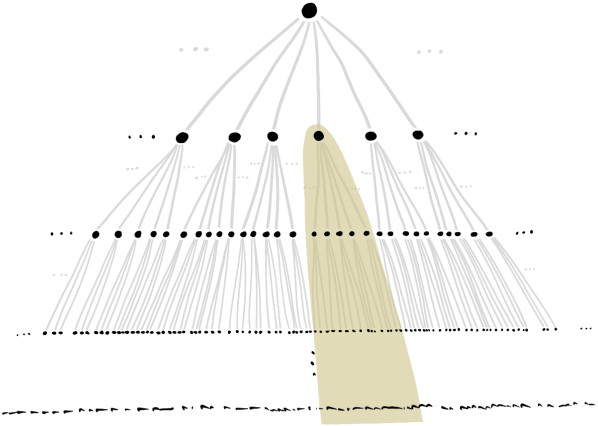
We present an easier-to-explain example with similar behaviour.

The singular ideal – a minimal and effective example



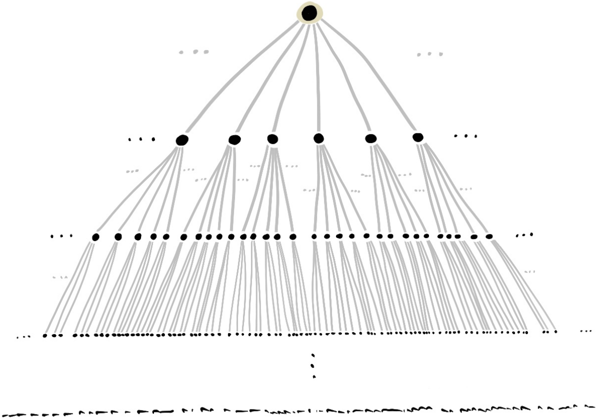
$\mathcal{G}^{(0)}$: black dots above (in bijection with finite and infinite rays in the tree)

The singular ideal – a minimal and effective example



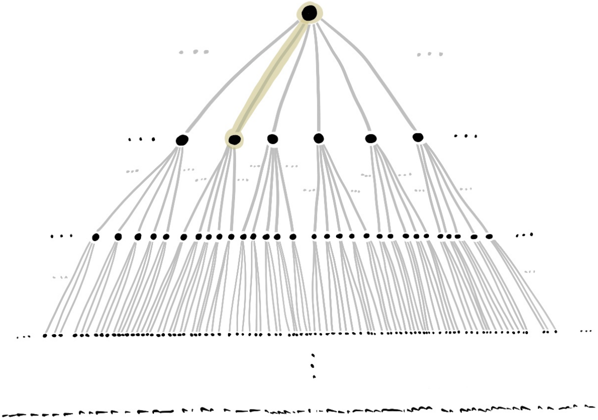
Topology on $\mathcal{G}^{(0)}$: generated by the clopen 'cones'

The singular ideal – a minimal and effective example



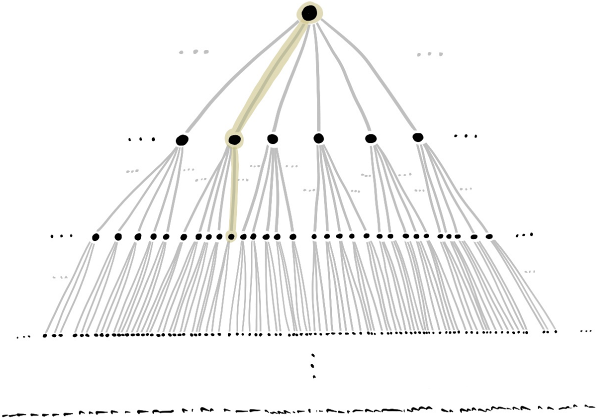
Convergence in $\mathcal{G}^{(0)}$:

The singular ideal – a minimal and effective example



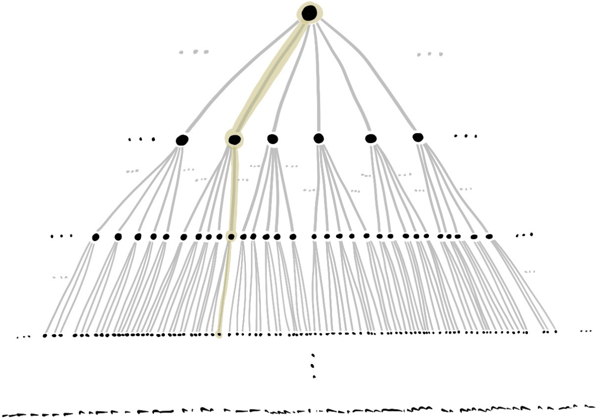
Convergence in $\mathcal{G}^{(0)}$:

The singular ideal – a minimal and effective example



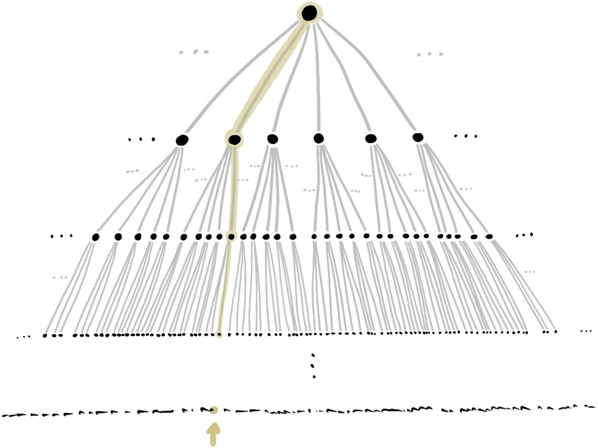
Convergence in $\mathcal{G}^{(0)}$:

The singular ideal – a minimal and effective example



Convergence in $\mathcal{G}^{(0)}$:

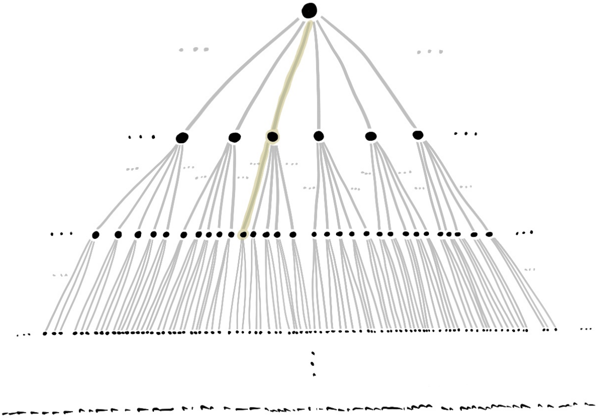
The singular ideal – a minimal and effective example



Convergence in $\mathcal{G}^{(0)}$:

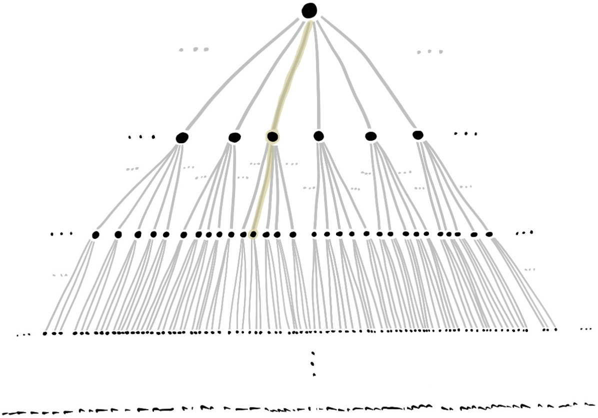
infinite rays are approximated by their finite prefixes

The singular ideal – a minimal and effective example



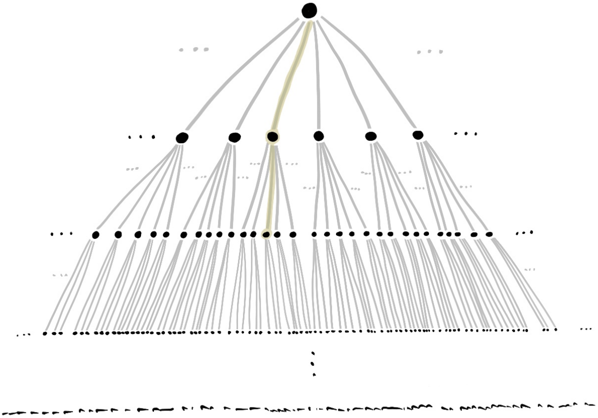
Convergence in $\mathcal{G}^{(0)}$:

The singular ideal – a minimal and effective example



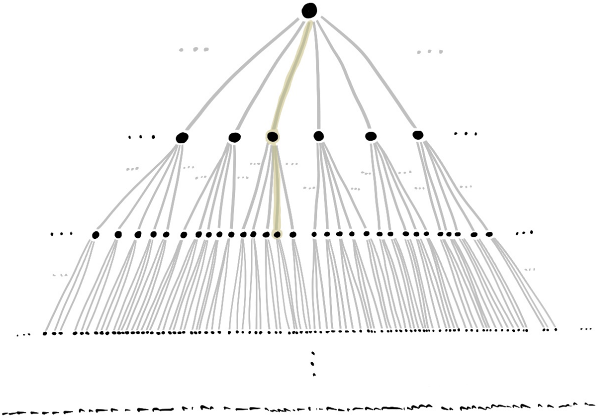
Convergence in $\mathcal{G}^{(0)}$:

The singular ideal – a minimal and effective example



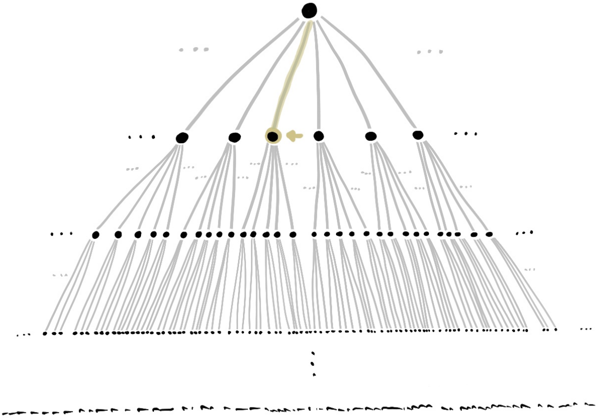
Convergence in $\mathcal{G}^{(0)}$:

The singular ideal – a minimal and effective example



Convergence in $\mathcal{G}^{(0)}$:

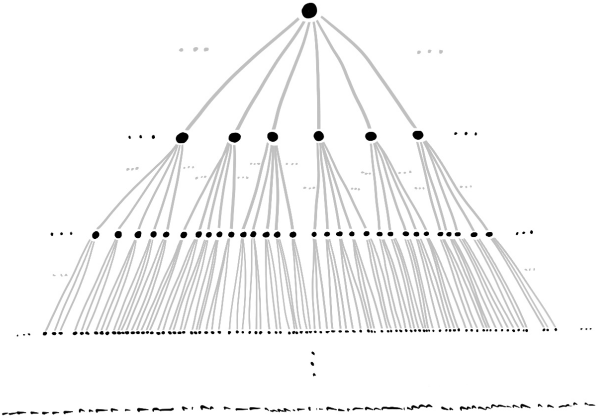
The singular ideal – a minimal and effective example



Convergence in $\mathcal{G}^{(0)}$:

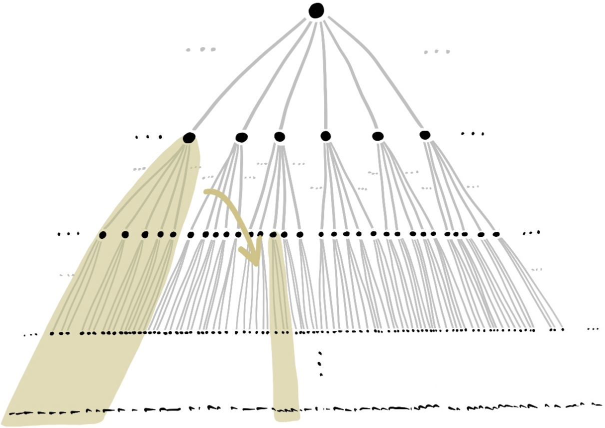
finite rays are approximated by rays with that common prefix

The singular ideal – a minimal and effective example



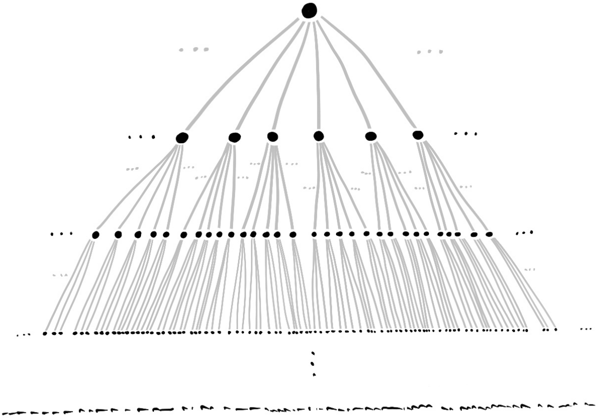
$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:**

The singular ideal – a minimal and effective example



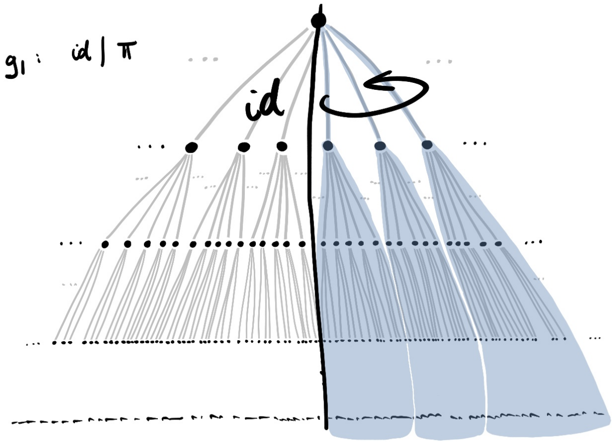
$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:** 'prefix exchange' maps – these ensure minimality

The singular ideal – a minimal and effective example



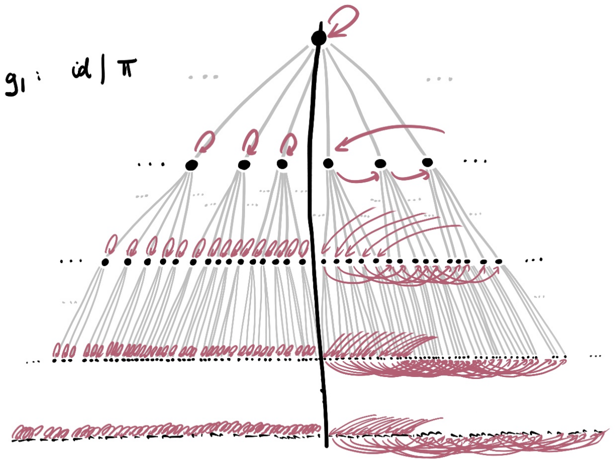
$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:** additional 'group' maps create a singular function

The singular ideal – a minimal and effective example



$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:**
the action of g_1

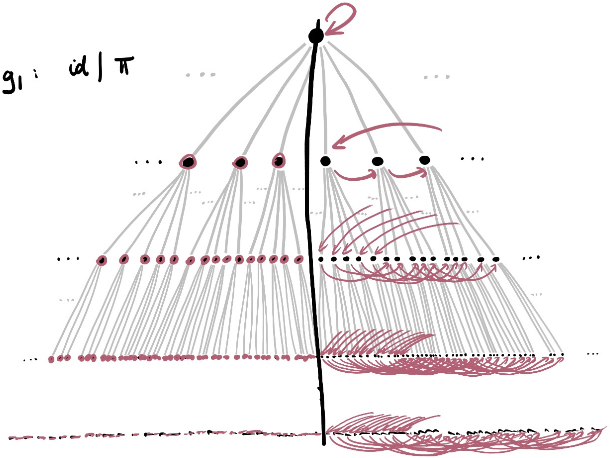
The singular ideal – a minimal and effective example



$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:**

the g_1-arrows in the transformation groupoid

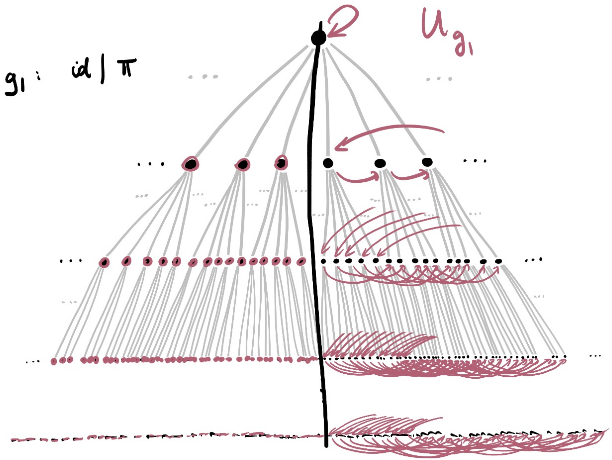
The singular ideal – a minimal and effective example



$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:**

the germs of the g_1-arrows

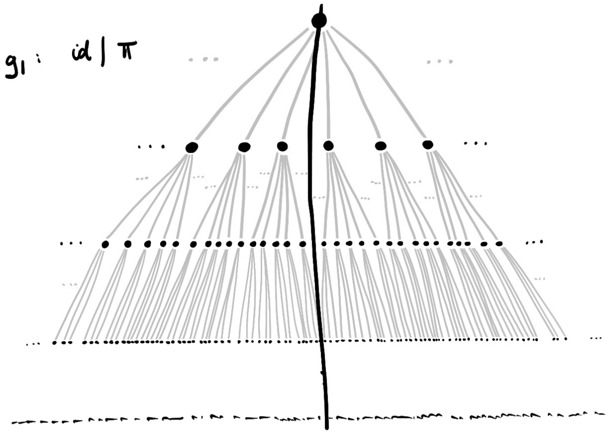
The singular ideal – a minimal and effective example



$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:**

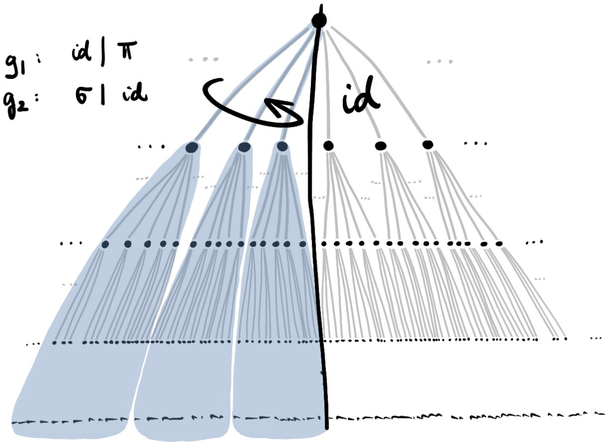
these form the compact open bisection U_{g_1}

The singular ideal – a minimal and effective example



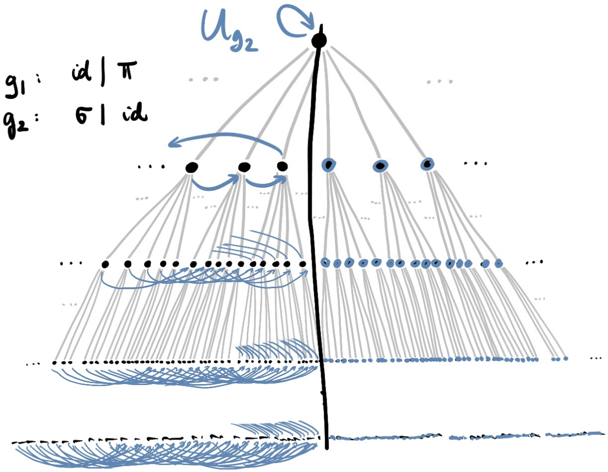
$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:**

The singular ideal – a minimal and effective example



$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:**
the action of g_2

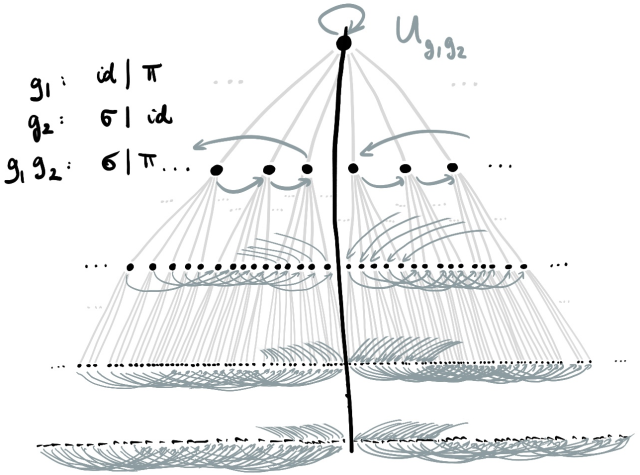
The singular ideal – a minimal and effective example



$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:**

we similarly obtain the compact open bisection U_{g_2}

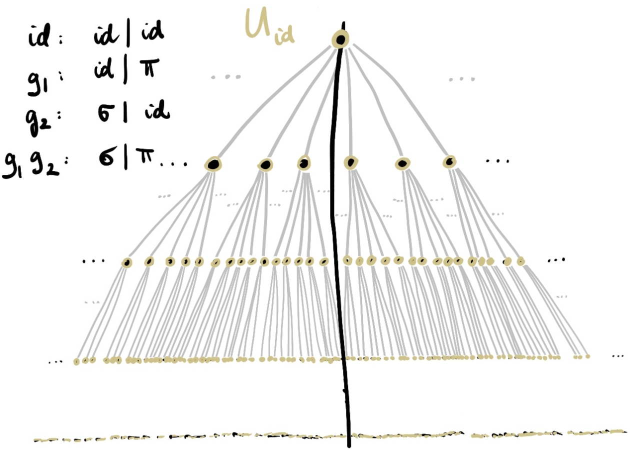
The singular ideal – a minimal and effective example



$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:**

the product $g_1 g_2$ gives $U_{g_1 g_2}$

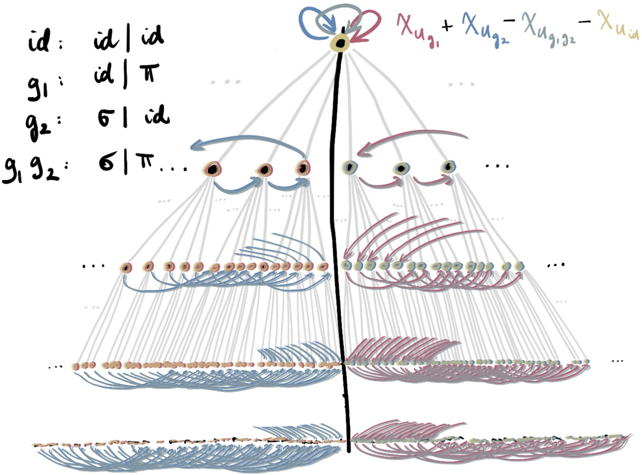
The singular ideal – a minimal and effective example



$\mathcal{G}$ is a groupoid of germs of an inverse semigroup. **The action:**

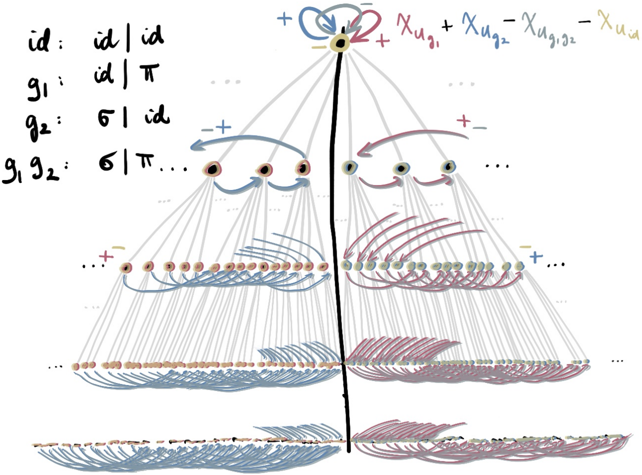
$\mathcal{G}^{(0)}$ itself is a compact open bisection U_{id}

The singular ideal – a minimal and effective example



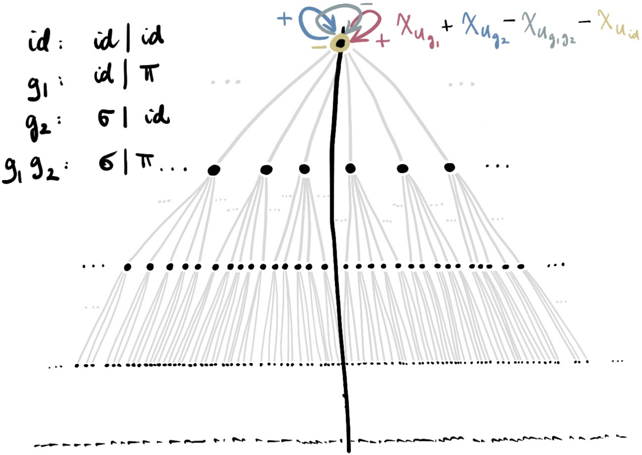
Consider the function $\chi_{U_{g_1}} + \chi_{U_{g_2}} - \chi_{U_{g_1 g_2}} - \chi_{U_{\mathrm{id}}}$

The singular ideal – a minimal and effective example



Consider the function $\chi_{U_{g_1}} + \chi_{U_{g_2}} - \chi_{U_{g_1 g_2}} - \chi_{U_{\mathrm{id}}}$

The singular ideal – a minimal and effective example



Consider the function $\chi_{U_{g_1}} + \chi_{U_{g_2}} - \chi_{U_{g_1 g_2}} - \chi_{U_{\mathrm{id}}} \in J_{alg}$

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Non-Hausdorff, second countable case:

Clark-Exel-Pardo-Starling-Sims [Trans. AMS, 2019]

$\iff$ $\mathcal{G}$ is minimal, effective, and $J_{alg}(\mathcal{G}) = 0$

$\Longleftarrow$ $\mathcal{G}$ is minimal, effective, $J = 0$
$\Longrightarrow$ $\mathcal{G}$ is minimal, effective and $J = 0$ when $\mathcal{G}$ is amenable

Where $J = \{\mathfrak{f} \in C^*_{red}(\mathcal{G}) : \mathfrak{f} \text{ vanishes on a dense subset}\}$ is called the (C^*-)singular ideal of $C^*_{red}(\mathcal{G})$.

$J_{alg} := J \cap \mathbb{C}\mathcal{G}$ is called the (algebraic) singular ideal of $\mathbb{C}\mathcal{G}$.

Simplicity

When is $\mathbb{C}\mathcal{G}$ simple (as a complex algebra)?

When is $C^*_{red}(\mathcal{G})$ simple (as a C^*-algebra)?

Non-Hausdorff, second countable case:

Clark-Exel-Pardo-Starling-Sims [Trans. AMS, 2019]

$\iff$ $\mathcal{G}$ is minimal, effective, and $J_{alg}(\mathcal{G}) = 0$

$\Longleftarrow$ $\mathcal{G}$ is minimal, effective, $J = 0$
$\Longrightarrow$ $\mathcal{G}$ is minimal, effective and $J = 0$ when $\mathcal{G}$ is amenable

Where $J = \{\mathfrak{f} \in C^*_{red}(\mathcal{G}) : \mathfrak{f}$ vanishes on a dense subset$\}$ is called the (C^*-)singular ideal of $C^*_{red}(\mathcal{G})$.

$J_{alg} := J \cap \mathbb{C}\mathcal{G}$ is called the (algebraic) singular ideal of $\mathbb{C}\mathcal{G}$.

It is natural to ask:
Question 1: Does $J_{alg} = 0$ imply $J = 0$?
Question 2: Is J_{alg} dense in J?

Progress

Question 1: Does $J_{alg} = 0$ imply $J = 0$?

Progress

Question 1: Does $J_{alg} = 0$ imply $J = 0$?

The question is still open but there are positive results:

Progress

Question 1: Does $J_{alg} = 0$ imply $J = 0$?

The question is still open but there are positive results:

- Gardella-Nekrashevych-Steinberg-Vdovina [J. Math. Phys, 2025]: Yes if $\mathcal{G}$ comes from a contracting self-similar group action

Progress

Question 1: Does $J_{alg} = 0$ imply $J = 0$?

The question is still open but there are positive results:

- Gardella-Nekrashevych-Steinberg-Vdovina [J. Math. Phys, 2025]: Yes if $\mathcal{G}$ comes from a contracting self-similar group action
- Brix-Gonzales-Hume-Li [preprint, 2025]: Yes if every net in $\mathcal{G}$ has finitely many limits

Progress

Question 1: Does $J_{alg} = 0$ imply $J = 0$?

The question is still open but there are positive results:

- Gardella-Nekrashevych-Steinberg-Vdovina [J. Math. Phys, 2025]: Yes if $\mathcal{G}$ comes from a contracting self-similar group action
- Brix-Gonzales-Hume-Li [preprint, 2025]: Yes if every net in $\mathcal{G}$ has finitely many limits (in particular, if $\mathcal{G}$ comes from a contracting self-similar group action)

Progress

Question 1: Does $J_{alg} = 0$ imply $J = 0$?

The question is still open but there are positive results:

- Gardella-Nekrashevych-Steinberg-Vdovina [J. Math. Phys, 2025]: Yes if $\mathcal{G}$ comes from a contracting self-similar group action
- Brix-Gonzales-Hume-Li [preprint, 2025]: Yes if every net in $\mathcal{G}$ has finitely many limits (in particular, if $\mathcal{G}$ comes from a contracting self-similar group action)
- Hume [preprint, 2025] obtains further sufficient conditions on the subgroups of $\mathcal{G}$, satisfied by e.g. all virtually solvable groups

Progress

Question 1: Does $J_{alg} = 0$ imply $J = 0$?

The question is still open but there are positive results:

- Gardella-Nekrashevych-Steinberg-Vdovina [J. Math. Phys, 2025]: Yes if $\mathcal{G}$ comes from a contracting self-similar group action
- Brix-Gonzales-Hume-Li [preprint, 2025]: Yes if every net in $\mathcal{G}$ has finitely many limits (in particular, if $\mathcal{G}$ comes from a contracting self-similar group action)
- Hume [preprint, 2025] obtains further sufficient conditions on the subgroups of $\mathcal{G}$, satisfied by e.g. all virtually solvable groups

Question 2: Is J_{alg} dense in J?

Progress

Question 1: Does $J_{alg} = 0$ imply $J = 0$?

The question is still open but there are positive results:

- Gardella-Nekrashevych-Steinberg-Vdovina [J. Math. Phys, 2025]: Yes if $\mathcal{G}$ comes from a contracting self-similar group action
- Brix-Gonzales-Hume-Li [preprint, 2025]: Yes if every net in $\mathcal{G}$ has finitely many limits (in particular, if $\mathcal{G}$ comes from a contracting self-similar group action)
- Hume [preprint, 2025] obtains further sufficient conditions on the subgroups of $\mathcal{G}$, satisfied by e.g. all virtually solvable groups

Question 2: Is J_{alg} dense in J?

- Gonzales-Hume [preprint, 2025]: Yes if every net in $\mathcal{G}$ has finitely many limits

Progress

Question 1: Does $J_{alg} = 0$ imply $J = 0$?

The question is still open but there are positive results:

- Gardella-Nekrashevych-Steinberg-Vdovina [J. Math. Phys, 2025]: Yes if $\mathcal{G}$ comes from a contracting self-similar group action
- Brix-Gonzales-Hume-Li [preprint, 2025]: Yes if every net in $\mathcal{G}$ has finitely many limits (in particular, if $\mathcal{G}$ comes from a contracting self-similar group action)
- Hume [preprint, 2025] obtains further sufficient conditions on the subgroups of $\mathcal{G}$, satisfied by e.g. all virtually solvable groups

Question 2: Is J_{alg} dense in J?

- Gonzales-Hume [preprint, 2025]: Yes if every net in $\mathcal{G}$ has finitely many limits, or if all subgroups of $\mathcal{G}$ are abelian

Progress

Question 1: Does $J_{alg} = 0$ imply $J = 0$?

The question is still open but there are positive results:

- Gardella-Nekrashevych-Steinberg-Vdovina [J. Math. Phys, 2025]: Yes if $\mathcal{G}$ comes from a contracting self-similar group action
- Brix-Gonzales-Hume-Li [preprint, 2025]: Yes if every net in $\mathcal{G}$ has finitely many limits (in particular, if $\mathcal{G}$ comes from a contracting self-similar group action)
- Hume [preprint, 2025] obtains further sufficient conditions on the subgroups of $\mathcal{G}$, satisfied by e.g. all virtually solvable groups

Question 2: Is J_{alg} dense in J?

- Gonzales-Hume [preprint, 2025]: Yes if every net in $\mathcal{G}$ has finitely many limits, or if all subgroups of $\mathcal{G}$ are abelian

But in general, the answer is NO, in fact there are even minimal and effective counterexamples (Martínez, Sz., [preprint, 2025]).

The strategy

Recall:

- $J_{alg} \lhd \mathbb{C}\mathcal{G} = \text{span}\{\chi_U : U \text{ is a compact open bisection}\}$;
- $J \lhd C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}} \subseteq \overline{\mathbb{C}\mathcal{G}}^{\infty}$;
- $\mathfrak{f} \in J \iff \text{supp}\,\mathfrak{f}$ contains no open set, $J_{alg} = J \cap \mathbb{C}$

The strategy

Recall:

- $J_{alg} \lhd \mathbb{C}\mathcal{G} = \text{span}\{\chi_U : U \text{ is a compact open bisection}\}$;
- $J \lhd C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}} \subseteq \overline{\mathbb{C}\mathcal{G}}^{\infty}$;
- $\mathfrak{f} \in J \iff \text{supp}\,\mathfrak{f}$ contains no open set, $J_{alg} = J \cap \mathbb{C}$

We construct a groupoid $\mathcal{G}$ where $\overline{J_{alg}} \subsetneq J$. Specifically, $\mathcal{G}$ will have:

The strategy

Recall:

- $J_{alg} \triangleleft \mathbb{C}\mathcal{G} = \text{span}\{\chi_U : U \text{ is a compact open bisection}\}$;
- $J \triangleleft C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}} \subseteq \overline{\mathbb{C}\mathcal{G}}^{\infty}$;
- $\mathfrak{f} \in J \iff \text{supp}\,\mathfrak{f}$ contains no open set, $J_{alg} = J \cap \mathbb{C}$

We construct a groupoid $\mathcal{G}$ where $\overline{J_{alg}} \subsetneq J$. Specifically, $\mathcal{G}$ will have:

- an open, but non-compact set $B \in \mathcal{G}^{(0)}$ such that $\chi_B \in C^*_{red}(\mathcal{G})$

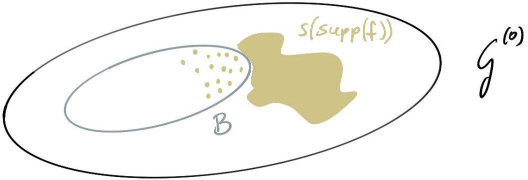
The strategy

Recall:

- $J_{alg} \lhd \mathbb{C}\mathcal{G} = \text{span}\{\chi_U : U \text{ is a compact open bisection}\}$;
- $J \lhd C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}} \subseteq \overline{\mathbb{C}\mathcal{G}}^{\infty}$;
- $\mathfrak{f} \in J \iff \text{supp}\,\mathfrak{f}$ contains no open set, $J_{alg} = J \cap \mathbb{C}$

We construct a groupoid $\mathcal{G}$ where $\overline{J_{alg}} \subsetneq J$. Specifically, $\mathcal{G}$ will have:

- an open, but non-compact set $B \in \mathcal{G}^{(0)}$ such that $\chi_B \in C^*_{red}(\mathcal{G})$
- a function $\mathfrak{f} \in \mathbb{C}\mathcal{G}$ such that $\mathfrak{f} * \chi_B = \mathfrak{f}|_{\mathcal{G}_B}$ is singular, but $\mathfrak{f}$ has large support outside $\mathcal{G}_B$



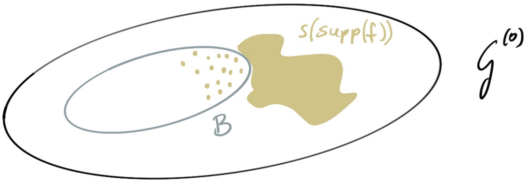
The strategy

Recall:

- $J_{alg} \lhd \mathbb{C}\mathcal{G} = \text{span}\{\chi_U : U \text{ is a compact open bisection}\}$;
- $J \lhd C^*_{red}(\mathcal{G}) = \overline{\mathbb{C}\mathcal{G}} \subseteq \overline{\mathbb{C}\mathcal{G}}^{\infty}$;
- $\mathfrak{f} \in J \iff \text{supp}\,\mathfrak{f}$ contains no open set, $J_{alg} = J \cap \mathbb{C}$

We construct a groupoid $\mathcal{G}$ where $\overline{J_{alg}} \subsetneq J$. Specifically, $\mathcal{G}$ will have:

- an open, but non-compact set $B \in \mathcal{G}^{(0)}$ such that $\chi_B \in C^*_{red}(\mathcal{G})$
- a function $\mathfrak{f} \in \mathbb{C}\mathcal{G}$ such that $\mathfrak{f} * \chi_B = \mathfrak{f}|_{\mathcal{G}_B}$ is singular (a B-singular function), but $\mathfrak{f}$ has large support outside $\mathcal{G}_B$



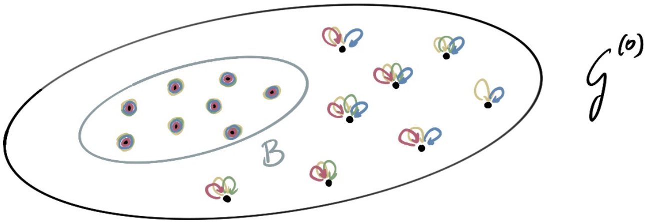
The strategy: creating χ_B

Our groupoid $\mathcal{G}$ will be a bundle of groups, i.e. $s(\gamma) = r(\gamma)$ for any $\gamma \in \mathcal{G}$.

The strategy: creating χ_B

Our groupoid $\mathcal{G}$ will be a bundle of groups, i.e. $s(\gamma) = r(\gamma)$ for any $\gamma \in \mathcal{G}$.

Suppose there exists a sequence $\{B_i\}_{i=1}^n$ of compact open bisections such that $B_i \cap B_j = B \subseteq \mathcal{G}^{(0)}$ for $i \neq j$.

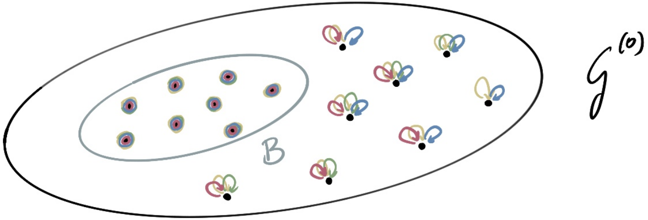


The strategy: creating χ_B

Our groupoid $\mathcal{G}$ will be a bundle of groups, i.e. $s(\gamma) = r(\gamma)$ for any $\gamma \in \mathcal{G}$.

Suppose there exists a sequence $\{B_i\}_{i=1}^n$ of compact open bisections such that $B_i \cap B_j = B \subseteq \mathcal{G}^{(0)}$ for $i \neq j$.

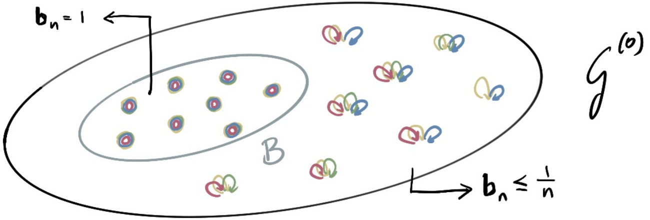
Define $\mathfrak{b}_n = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} \chi_{B_i}$.



Our groupoid $\mathcal{G}$ will be a bundle of groups, i.e. $s(\gamma) = r(\gamma)$ for any $\gamma \in \mathcal{G}$.

Suppose there exists a sequence $\{B_i\}_{i=1}^n$ of compact open bisections such that $B_i \cap B_j = B \subseteq \mathcal{G}^{(0)}$ for $i \neq j$.

Define $\mathfrak{b}_n = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \chi_{B_i}$.

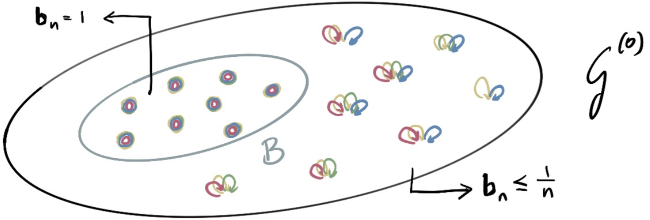


The strategy: creating χ_B

Our groupoid $\mathcal{G}$ will be a bundle of groups, i.e. $s(\gamma) = r(\gamma)$ for any $\gamma \in \mathcal{G}$.

Suppose there exists a sequence $\{B_i\}_{i=1}^n$ of compact open bisections such that $B_i \cap B_j = B \subseteq \mathcal{G}^{(0)}$ for $i \neq j$.

Define $\mathfrak{b}_n = \frac{1}{n} \sum_{i=1}^{n} \chi_{B_i} \xrightarrow{||\cdot||_\infty} \chi_B$.

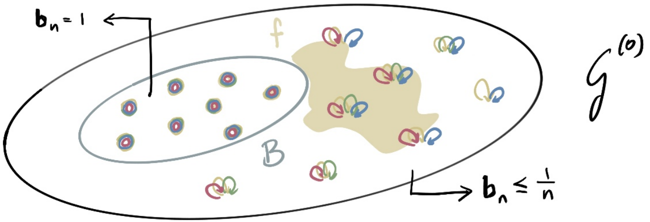


The strategy: creating χ_B

Our groupoid $\mathcal{G}$ will be a bundle of groups, i.e. $s(\gamma) = r(\gamma)$ for any $\gamma \in \mathcal{G}$.

Suppose there exists a sequence $\{B_i\}_{i=1}^n$ of compact open bisections such that $B_i \cap B_j = B \subseteq \mathcal{G}^{(0)}$ for $i \neq j$.

Define $\mathfrak{b}_n = \frac{1}{n} \sum_{i=1}^{n} \chi_{B_i} \xrightarrow{||\cdot||_\infty} \chi_B$.



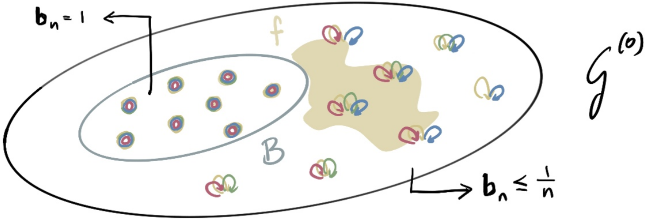
So $\mathfrak{f} * \mathfrak{b}_n \xrightarrow{||\cdot||_\infty} \mathfrak{f} * \chi_B$.

The strategy: creating χ_B

Our groupoid $\mathcal{G}$ will be a bundle of groups, i.e. $s(\gamma) = r(\gamma)$ for any $\gamma \in \mathcal{G}$.

Suppose there exists a sequence $\{B_i\}_{i=1}^n$ of compact open bisections such that $B_i \cap B_j = B \subseteq \mathcal{G}^{(0)}$ for $i \neq j$.

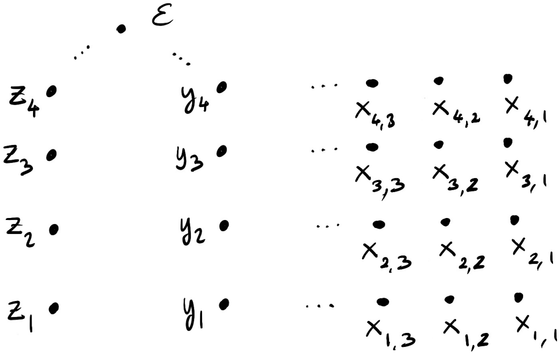
Define $\mathfrak{b}_n = \dfrac{1}{n}\sum_{i=1}^{n} \chi_{B_i} \xrightarrow{||\cdot||_\infty} \chi_B$.



So $\mathfrak{f} * \mathfrak{b}_n \xrightarrow{||\cdot||_\infty} \mathfrak{f} * \chi_B$.

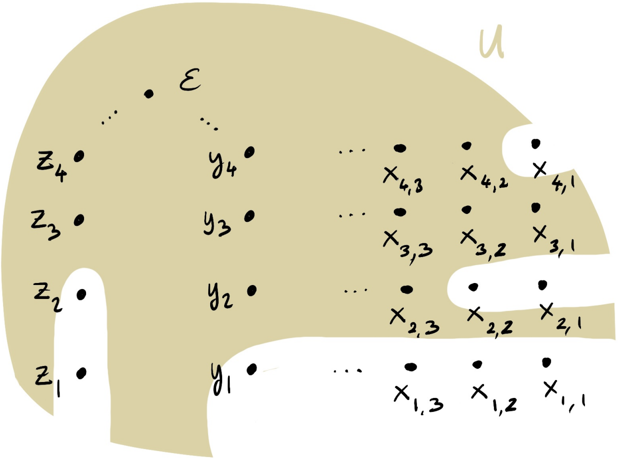
Convergence in C^*-norm comes at the cost of amenability.

The example: the definition of $\mathcal{G}$

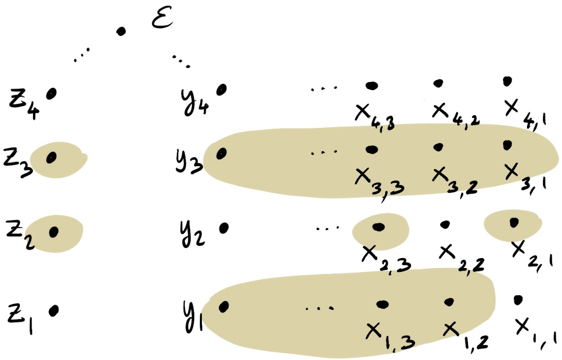


$\mathcal{G}^{(0)}$: black dots above (where $\cdots$ denotes convergence)

The example: the definition of $\mathcal{G}$

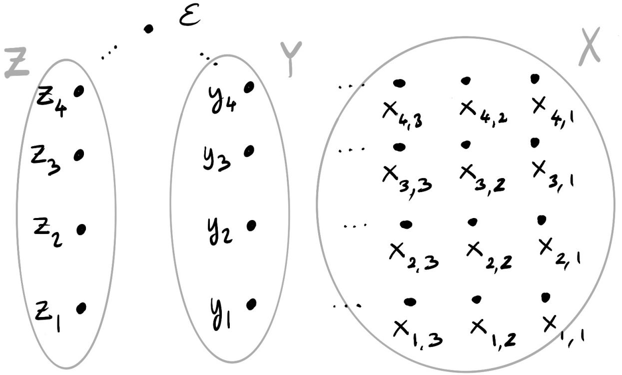


A 'typical' basic compact open neighborhood of ϵ



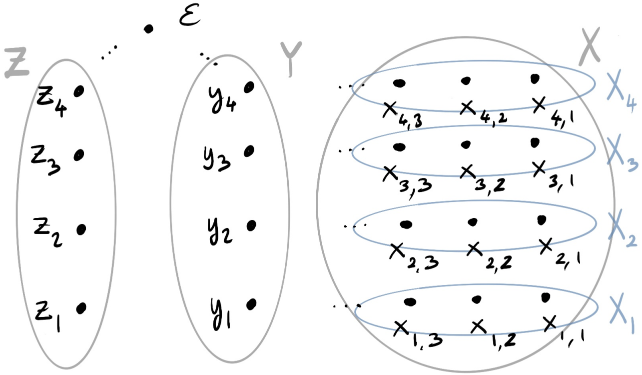
A ‘typical’ basic compact open set not containing ϵ

The example: the definition of $\mathcal{G}$



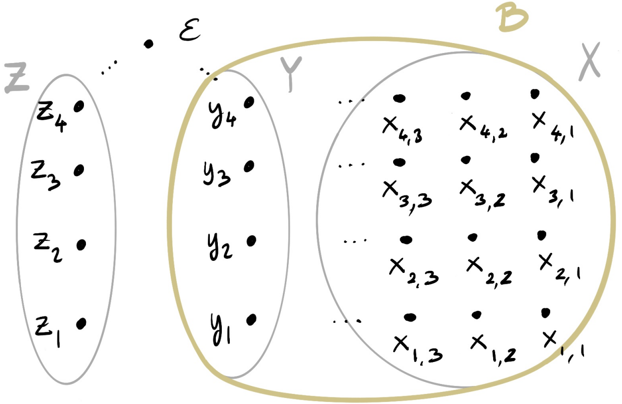
Notation: X, Y, Z

The example: the definition of $\mathcal{G}$



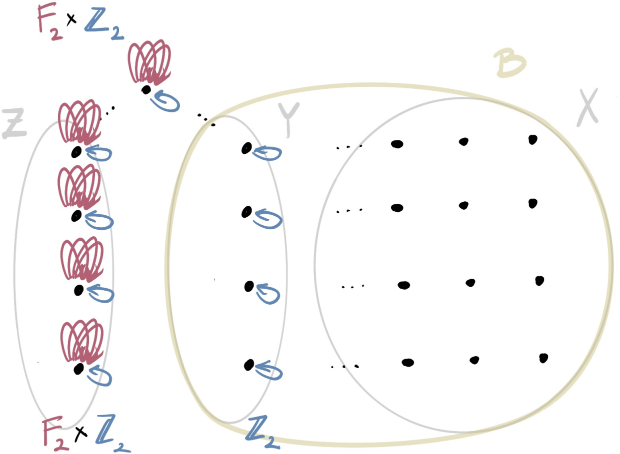
Notation: X, Y, Z

The example: the definition of $\mathcal{G}$



Notation: X, Y, Z and B: the set we 'scatter' to

The example: the definition of $\mathcal{G}$



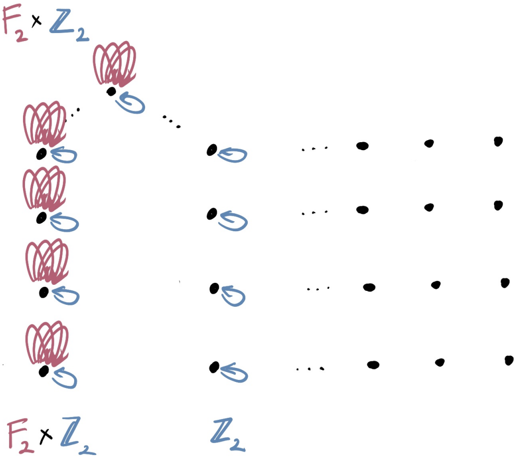
The groupoid:

units in Z and ϵ have the isotropy group $F_2 \times \mathbb{Z}_2$

units in Y have the isotropy group $\mathbb{Z}_2$

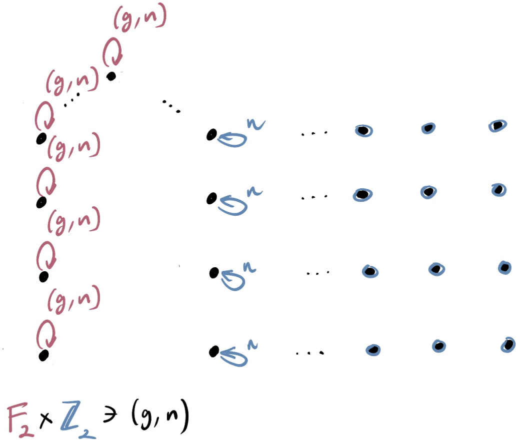
units in X have trivial isotropy group

The example: the definition of $\mathcal{G}$



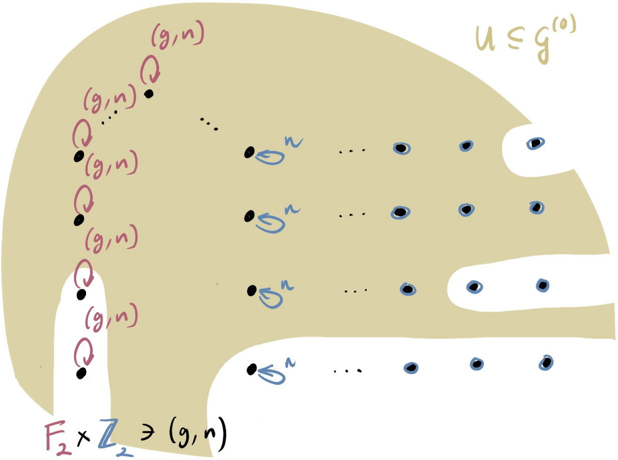
Basic compact open bisections:

The example: the definition of $\mathcal{G}$



Basic compact open bisections:

Every pair $(g, n) \in F_2 \times \mathbb{Z}_2$ defines a compact open bisection

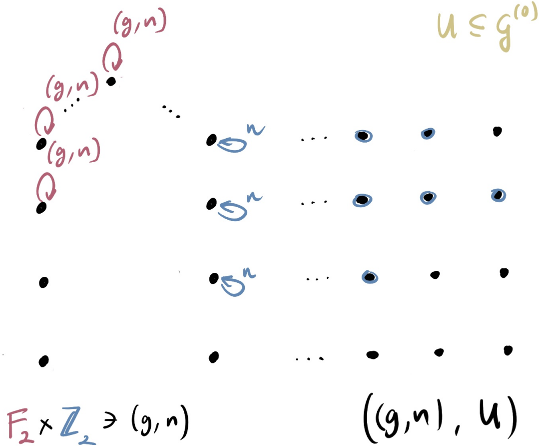


Basic compact open bisections:

Every pair $(g, n) \in F_2 \times \mathbb{Z}_2$ defines a compact open bisection

...which we can further restrict to any basic compact open $U \subseteq \mathcal{G}^{(0)}$

The example: the definition of $\mathcal{G}$

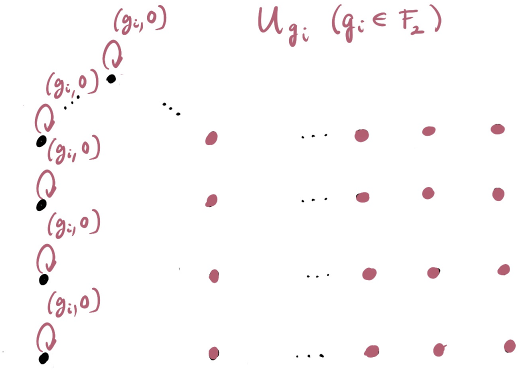


Basic compact open bisections:

Every pair $(g, n) \in F_2 \times \mathbb{Z}_2$ defines a compact open bisection

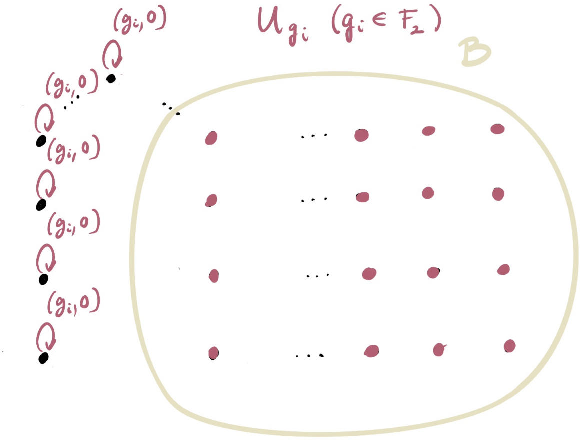
...which we can further restrict to any basic compact open $U \subseteq \mathcal{G}^{(0)}$

The example: the singular element of $C^*_{red}(\mathcal{G})$



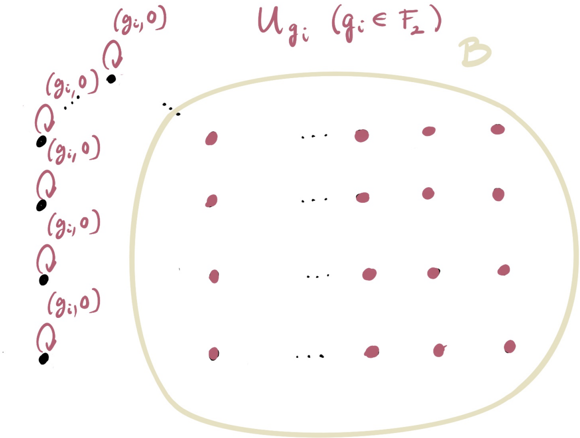
Let $g_i \in F_2, i \in \mathbb{N}$ be pairwise different.
Consider the compact open bisection $B_i = U_{g_i} = ((g_i, 0), \mathcal{G}^{(0)})$.

The example: the singular element of $C^*_{red}(\mathcal{G})$



We 'scatter' to B: $\mathfrak{b}_n = \frac{1}{n}\sum_{i=1}^{n} \chi_{U_{g_i}} \xrightarrow{\|\cdot\|_\infty} \chi_B$

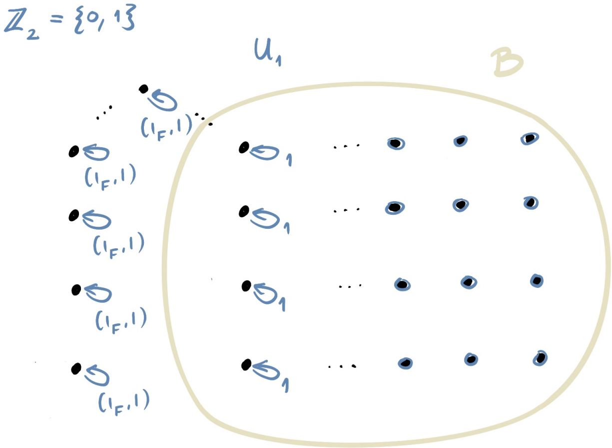
The example: the singular element of $C^*_{red}(\mathcal{G})$



We 'scatter' to B: $\mathfrak{b}_n = \frac{1}{n}\sum_{i=1}^{n} \chi_{U_{g_i}} \xrightarrow{||\cdot||_\infty} \chi_B$

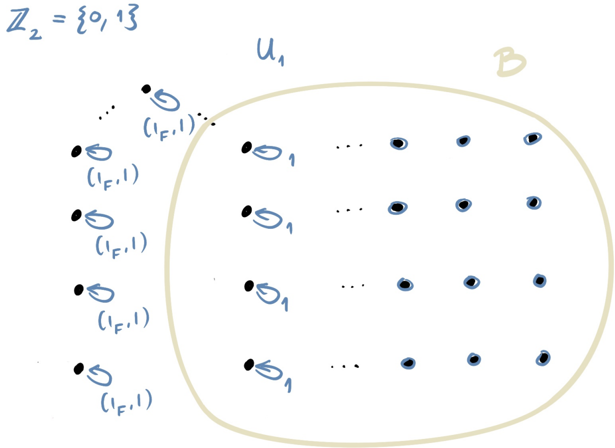
Convergence in norm is harder and requires a more careful choice of g_is.

The example: the singular element of $C^*_{red}(\mathcal{G})$



We create a B-singular function:

The example: the singular element of $C^*_{red}(\mathcal{G})$

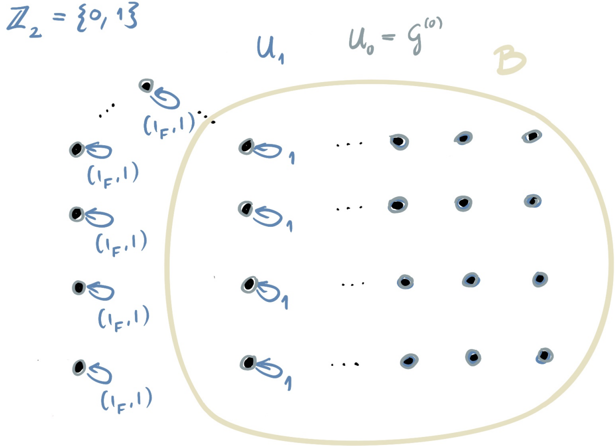


We create a B-singular function:

Let $U_1 = ((1_F, 1), \mathcal{G}^{(0)})$ compact open bisection

Let $U_0 = \mathcal{G}^{(0)}$ compact open bisection

The example: the singular element of $C^*_{red}(\mathcal{G})$

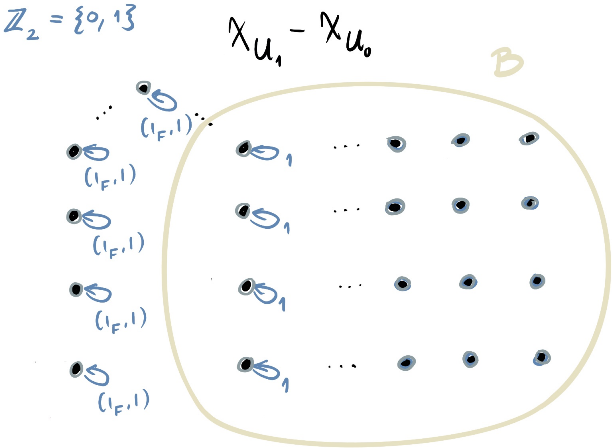


We create a B-singular function:

Let $U_1 = ((1_F, 1), \mathcal{G}^{(0)})$ compact open bisection

Let $U_0 = \mathcal{G}^{(0)}$ compact open bisection

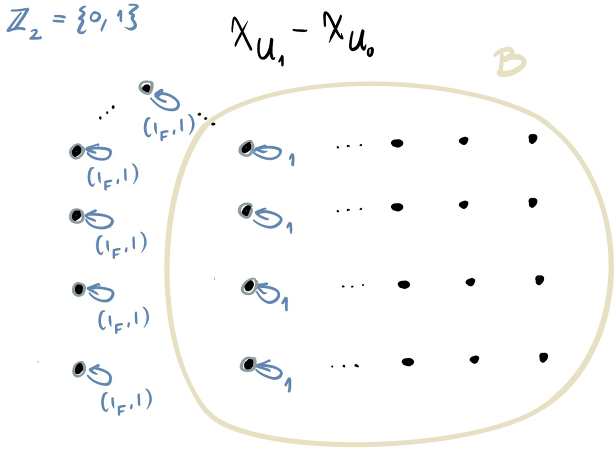
The example: the singular element of $C^*_{red}(\mathcal{G})$



We create a B-singular function:

$$\mathfrak{f} = \chi_{U_1} - \chi_{U_0}$$

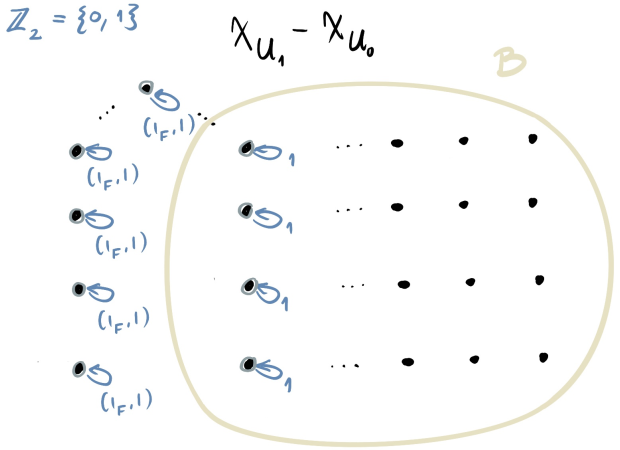
The example: the singular element of $C^*_{red}(\mathcal{G})$



We create a B-singular function:

$$\mathfrak{f} = \chi_{U_1} - \chi_{U_0}$$

The example: the singular element of $C^*_{red}(\mathcal{G})$

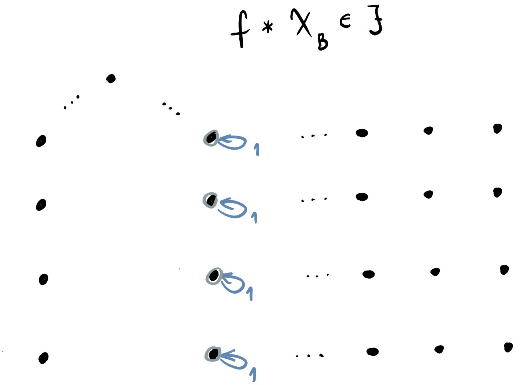


We create a B-singular function:

$$\mathfrak{f} = \chi_{U_1} - \chi_{U_0}$$

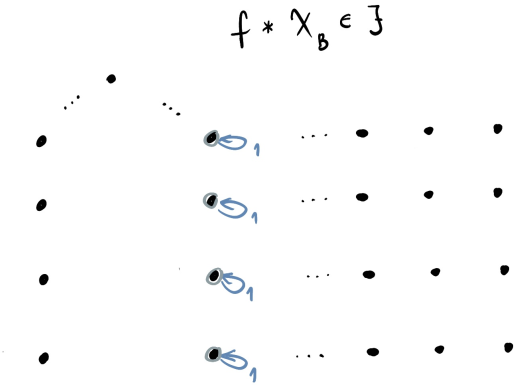
$\text{supp}(\mathfrak{f})|_{\mathcal{G}_B}$ contains no open sets $\implies \mathfrak{f} * \chi_B \in J$

The example: $f * \chi_B \notin \overline{J_{alg}}$



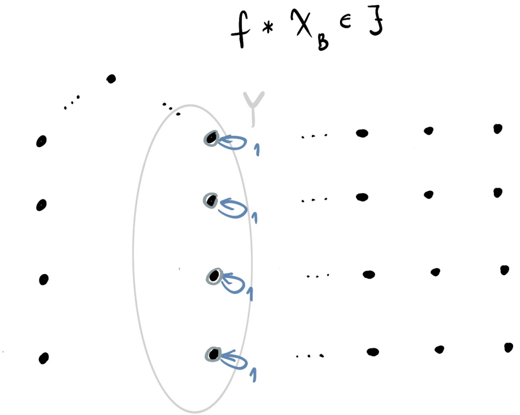
What remains is to show that $f * \chi_B \notin \overline{J_{alg}}$.

The example: $f * \chi_B \notin \overline{J_{alg}}$



We show something stronger: $f * \chi_B \notin \overline{J_{alg}}^{\infty}$.

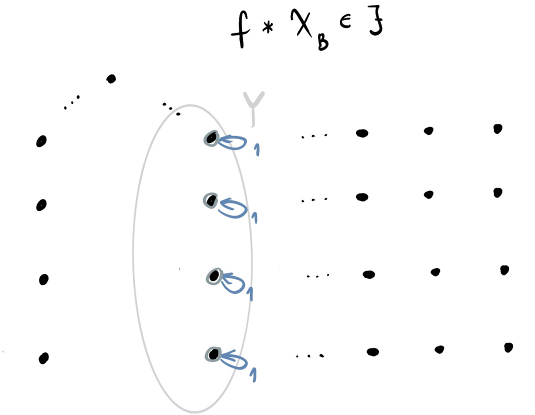
The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



We show something stronger: $f * \chi_B \notin \overline{J_{alg}}^{\infty}$.

Observe that if $||\mathfrak{f} - \mathfrak{g}||_\infty < 1$ then $Y \in \operatorname{supp} \mathfrak{g}$.

The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$

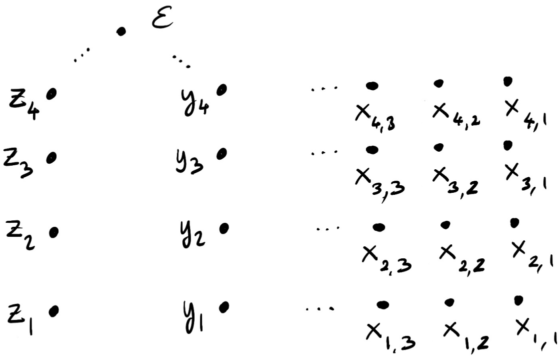


We show something stronger: $f * \chi_B \notin \overline{J_{alg}}^{\infty}$.

Observe that if $||\mathfrak{f} - \mathfrak{g}||_\infty < 1$ then $Y \in \operatorname{supp} \mathfrak{g}$.

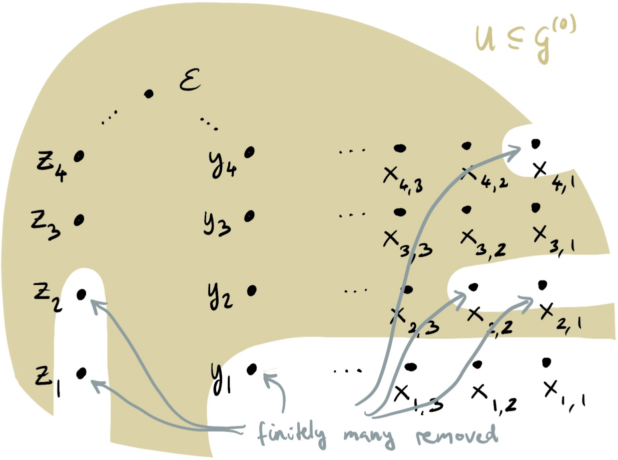
Claim: if $Y \subseteq \operatorname{supp} \mathfrak{g}$ for $\mathfrak{g} \in \mathbb{C}\mathcal{G}$, then $\mathfrak{g} \notin J_{alg}$.

The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



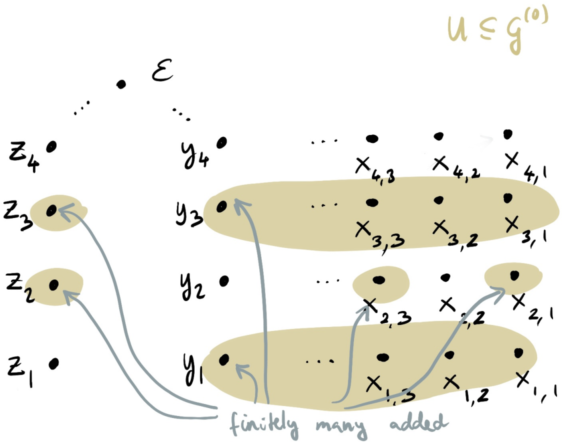
Let $\mathfrak{g} = \sum_{V \in F} c_V \chi_V$ where F is a finite set of compact open bisections, and suppose $Y \subseteq \operatorname{supp} \mathfrak{g}$.

The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



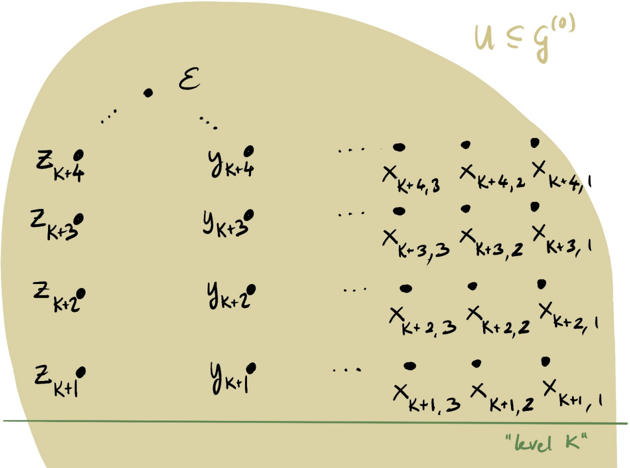
Any compact open bisection $V \in F$ is of the form $((g, n), U)$ where
(1) U is either G^0 with finitely many sets $\{x_{i,j}\}, \{y_k\} \cup X_k, \{z_l\}$ 'removed'

The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



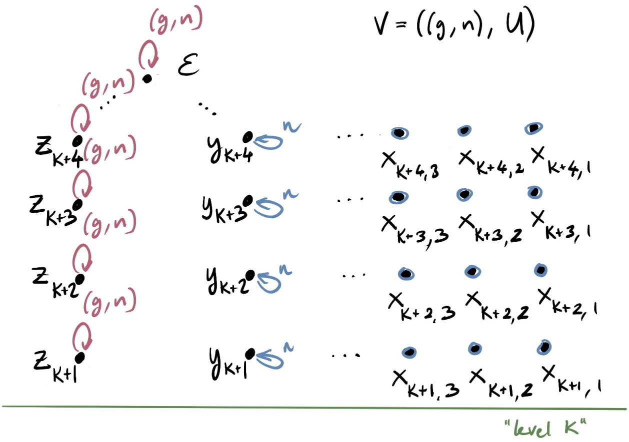
Any compact open bisection $V \in F$ is of the form $((g, n), U)$ where
(1) U is either G^0 with finitely many sets $\{x_{i,j}\}, \{y_k\} \cup X_k, \{z_l\}$ 'removed'
(2) or U consists of finitely many sets $\{x_{i,j}\}, \{y_k\} \cup X_k, \{z_l\}$ 'added'

The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



Since F is finite, there exists some $K \in \mathbb{N}$ larger than any 'added' or 'removed' indices

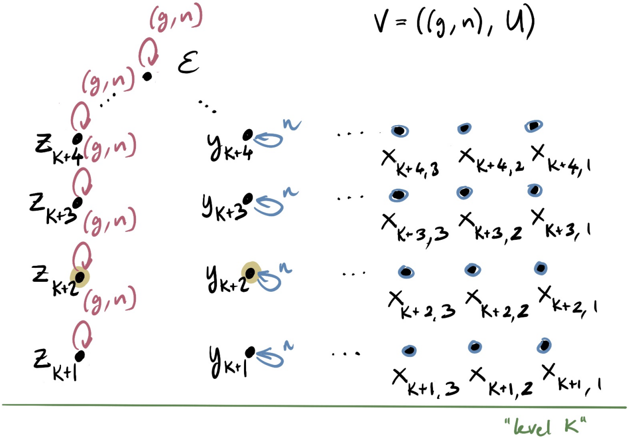
The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



Since F is finite, there exists some $K \in \mathbb{N}$ larger than any 'added' or 'removed' indices

Above the 'Kth level', any $V \in F$ looks either empty (type (ii)) or a like a full bisection (type(i)).

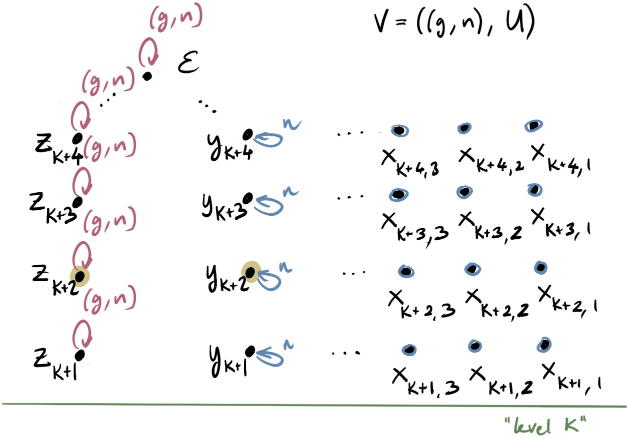
The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



Recall: $\mathfrak{g} = \sum_{V \in F} c_V \chi_V$. Let $i \geq K$. Then

$0 \neq \mathfrak{g}(y_i)$

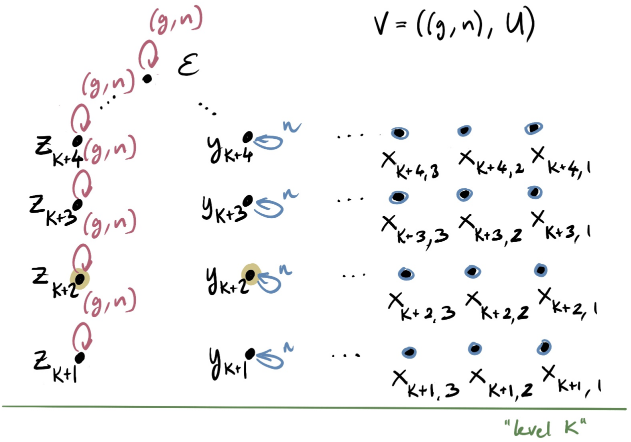
The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



Recall: $\mathfrak{g} = \sum_{V \in F} c_V \chi_V$. Let $i \geq K$. Then

$$0 \neq \mathfrak{g}(y_i) = \sum_{\substack{V=((g,0),U) \\ g \in F_2, U \text{ type } (i)}} c_V = \sum_{g \in F_2} \sum_{\substack{V=((g,0),U) \\ U \text{ type } (i)}} c_V$$

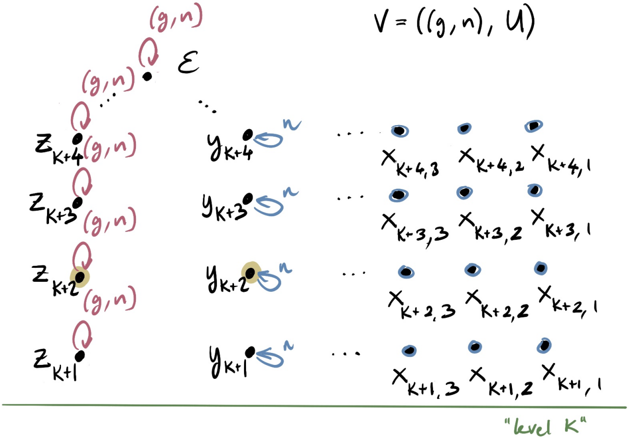
The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



Recall: $\mathfrak{g} = \sum_{V \in F} c_V \chi_V$. Let $i \geq K$. Then

$$0 \neq \mathfrak{g}(y_i) = \sum_{\substack{V=((g,0),U) \\ g \in F_2, U \text{ type } (i)}} c_V = \sum_{g \in F_2} \sum_{\substack{V=((g,0),U) \\ U \text{ type } (i)}} c_V = \sum_{g \in F_2} \mathfrak{g}((g,0), z_i)$$

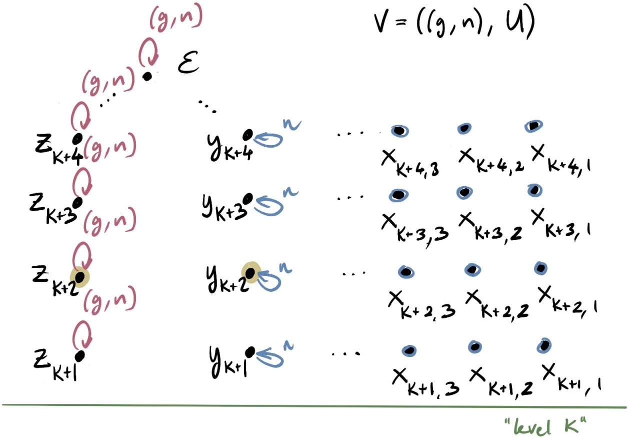
The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



Recall: $\mathfrak{g} = \sum_{V \in F} c_V \chi_V$. Let $i \geq K$. Then

$\implies$ there exists some g such that $\mathfrak{g}((g, 0), z_i) \neq 0$

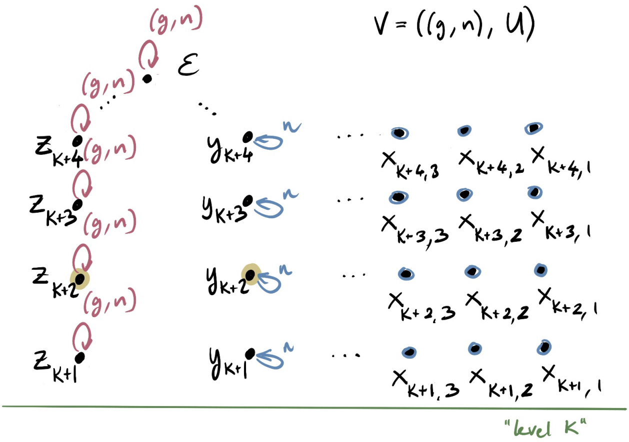
The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



Recall: $\mathfrak{g} = \sum_{V \in F} c_V \chi_V$. Let $i \geq K$. Then

$\implies$ there exists some g such that $\mathfrak{g}((g, 0), z_i) \neq 0 \implies \mathfrak{g} \notin J_{alg}$.

The example: $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$



Recall: $\mathfrak{g} = \sum_{V \in F} c_V \chi_V$. Let $i \geq K$. Then

$\implies$ there exists some g such that $\mathfrak{g}((g,0), z_i) \neq 0 \implies \mathfrak{g} \notin J_{alg}$.

This completes the proof that $\mathfrak{f} * \chi_B \notin \overline{J_{alg}}$.

Remarks

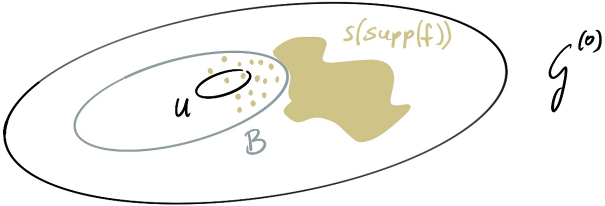
- We also have a minimal and effective counterexample: this is the groupoid of germs of an (inverse semigroup) action on an infinitary tree. The basic strategy is the same.

Remarks

- We also have a minimal and effective counterexample: this is the groupoid of germs of an (inverse semigroup) action on an infinitary tree. The basic strategy is the same.
- Our strategy never gives $J_{alg} = 0$:

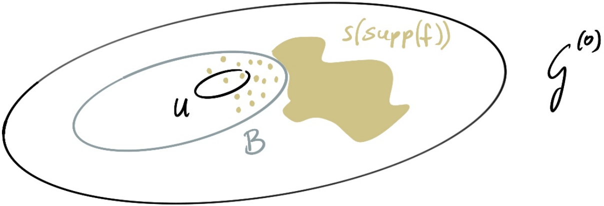
Remarks

- We also have a minimal and effective counterexample: this is the groupoid of germs of an (inverse semigroup) action on an infinitary tree. The basic strategy is the same.
- Our strategy never gives $J_{alg} = 0$: there is always a compact open $U \subseteq B$ intersecting $s(\operatorname{supp} \mathfrak{f})$, so $\operatorname{supp} \mathfrak{f} * \chi_U \in J_{alg}$.



Remarks

- We also have a minimal and effective counterexample: this is the groupoid of germs of an (inverse semigroup) action on an infinitary tree. The basic strategy is the same.
- Our strategy never gives $J_{alg} = 0$: there is always a compact open $U \subseteq B$ intersecting $s(\operatorname{supp} \mathfrak{f})$, so $\operatorname{supp} \mathfrak{f} * \chi_U \in J_{alg}$.



In particular, this is still open:
Question 1: Does $J_{alg} = 0$ imply $J = 0$?